REVIEW OF LECTURES 09-12

08/06/2013

ECE225 CIRCUIT ANALYSIS
Summary of Chapter 5 (Parts 2 & 3)

RC circuit response (sudden change at $t = 0$)
$$v_C(t) = v_C(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$\tau = RC$$

RL circuit response (sudden change at $t = 0$)
$$i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)]e^{-t/\tau}$$
$$\tau = L/R$$

Op-amp integrator
$$v_{out}(t) = -\frac{1}{RC} \int_{t_0}^{t} v_i \, dt + v_{out}(t_0)$$

Op-amp differentiator
$$v_{out}(t) = -RC \frac{dv_i}{dt}$$
RC and RL First-Order Circuits

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Objectives

Learn to:

- Use mathematical functions to describe several types of nonperiodic waveforms.
- Define the electrical properties of a capacitor, including its $i - v$ relationship and energy equation.
- Combine multiple capacitors when connected in series or in parallel.
- Define the electrical properties of an inductor, including its $i - v$ relationship and energy equation.
- Combine multiple inductors when connected in series or in parallel.

Capacitors ($C$) and inductors ($L$) are energy storage devices, in contrast with resistors, which are energy dissipation devices. This chapter examines the behavior of RC and RL circuits, to be followed in Chapter 6 with an examination of RLC circuits.

- Analyze the transient responses of RC and RL circuits.
- Design RC op-amp circuits to perform differentiation and integration and related operations.
- Apply Multisim to analyze RC and RL circuits.
### Table 5-4: Basic properties of $R$, $L$, and $C$.

<table>
<thead>
<tr>
<th>Property</th>
<th>$R$</th>
<th>$L$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$–$v$ relation</td>
<td>$i = \frac{v}{R}$</td>
<td>$i = \frac{1}{L} \int_{t_0}^{t} v , dt + i(t_0)$</td>
<td>$i = C \frac{dv}{dt}$</td>
</tr>
<tr>
<td>$v$–$i$ relation</td>
<td>$v = iR$</td>
<td>$v = L \frac{di}{dt}$</td>
<td>$v = \frac{1}{C} \int_{t_0}^{t} i , dt + v(t_0)$</td>
</tr>
<tr>
<td>$p$ (power transfer in)</td>
<td>$p = i^2R$</td>
<td>$p = Li \frac{di}{dt}$</td>
<td>$p = Cv \frac{dv}{dt}$</td>
</tr>
<tr>
<td>$w$ (stored energy)</td>
<td>$0$</td>
<td>$w = \frac{1}{2}Li^2$</td>
<td>$w = \frac{1}{2}Cv^2$</td>
</tr>
<tr>
<td>Series combination</td>
<td>$R_{eq} = R_1 + R_2$</td>
<td>$L_{eq} = L_1 + L_2$</td>
<td>$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$</td>
</tr>
<tr>
<td>Parallel combination</td>
<td>$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$</td>
<td>$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$</td>
<td>$C_{eq} = C_1 + C_2$</td>
</tr>
<tr>
<td>dc behavior</td>
<td>no change</td>
<td>short circuit</td>
<td>open circuit</td>
</tr>
<tr>
<td>Can $v$ change instantaneously?</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Can $i$ change instantaneously?</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>
Response Terminology

**Source dependence**

Natural response – response in absence of sources

Forced response – response due to external source

Complete response = Natural + Forced

**Time dependence**

Transient response – time-varying response (temporary)

Steady state response – time-independent or periodic (permanent)

Complete response = Transient + Steady State
Natural Response of Charged Capacitor

(a) $t = 0^-$ is the instant just before the switch is moved from terminal 1 to terminal 2.
(b) $t = 0$ is the instant just after it was moved; $t = 0$ is synonymous with $t = 0^+$ since the voltage across the capacitor cannot change instantaneously, it follows that

$$v(0) = v(0^-) = V_s.$$  

For $t \geq 0$, application of KVL to the loop in Fig. 5-28(c) gives

$$R i + v = 0 \quad \text{(for } t \geq 0), \quad (5.68)$$

where $i$ is the current through and $v$ is the voltage across the capacitor. Since $i = C \frac{dv}{dt}$,

$$RC \frac{dv}{dt} + v = 0. \quad (5.69)$$

Upon dividing both terms by $RC$, Eq. (5.69) takes the form

$$\frac{dv}{dt} + av = 0 \quad \text{(source-free),} \quad (5.70)$$

where

$$a = \frac{1}{RC}. \quad (5.71)$$
The standard procedure for solving Eq. (5.70) starts by multiplying both sides by $e^{at}$,

$$\frac{dv}{dt} e^{at} + ave^{at} = 0. \quad (5.72)$$

Next, we recognize that the sum of the two terms on the left-hand side is equal to the expansion of the differential of $(ve^{at})$,

$$\frac{d}{dt}(ve^{at}) = \frac{dv}{dt} e^{at} + ave^{at}. \quad (5.73)$$

Hence, Eq. (5.72) becomes

$$\frac{d}{dt}(ve^{at}) = 0. \quad (5.74)$$

Integrating both sides, we have

$$\int_0^t \frac{d}{dt}(ve^{at}) \, dt = 0, \quad (5.75)$$

Performing the integration gives

$$ve^{at}|_0^t = 0$$

or

$$v(t) e^{at} - v(0) = 0. \quad (5.76)$$

Solving for $v(t)$, we have

$$v(t) = v(0) e^{-at},$$

$$= v(0) e^{-t/RC} \quad \text{(for } t \geq 0), \quad (5.77)$$

$$v(t) = v(0) e^{-t/\tau} \quad \text{(natural response)},$$

with

$$\tau = RC \quad \text{(s)},$$

which is called the **time constant** of the circuit.
Natural Response of Charged Capacitor

\[ i(t) = C \frac{dV}{dt} = C \frac{d}{dt} (V_s e^{-t/\tau}) = -C \frac{V_s}{\tau} e^{-t/\tau} \quad \text{(for } t \geq 0) \]

which simplifies to

\[ i(t) = -\frac{V_s}{R} e^{-t/\tau} u(t) \quad \text{(for } t \geq 0) \]

(natural response).

\[ p(t) = iV = -\frac{V_s}{R} e^{-t/\tau} \times V_s e^{-t/\tau} = -\frac{V_s^2}{R} e^{-2t/\tau} \quad \text{(for } t \geq 0). \]
General Response of RC Circuit

\[ v(0) = v(0^-) = V_{s1}. \]  \hspace{1cm} (5.86)

For \( t \geq 0 \), the voltage equation for the loop in Fig. 5-30(c) is

\[ -V_{s2} + i R + v = 0. \]  \hspace{1cm} (5.87)

Upon using \( i = C \frac{dv}{dt} \) and rearranging its terms, Eq. (5.87) can be written in the differential-equation form

\[ \frac{dv}{dt} + av = b, \]  \hspace{1cm} (5.88)

where

\[ a = \frac{1}{RC} \quad \text{and} \quad b = \frac{V_{s2}}{RC}. \]  \hspace{1cm} (5.89)
Solution of

\[ \frac{dv}{dt} + av = b, \]

\[ \frac{d}{dt}(ve^{at}) = be^{at}. \]

Integrating both sides,

\[ \int_0^t \frac{d}{dt}(ve^{at}) \, dt = \int_0^t be^{at} \]

gives

\[ ve^{at} \bigg|_0^t = \frac{b}{a} e^{at} \bigg|_0^t. \]

Upon evaluating the functions at the two limits, we have

\[ v(t) e^{at} - v(0) = \frac{b}{a} e^{at} - \frac{b}{a}, \]

and then solving for \( v(t) \), we have

\[ v(t) = v(0) e^{-at} + \frac{b}{a} (1 - e^{-at}). \]

As \( t \to \infty \), \( v(t) \) reduces to

\[ v(\infty) = \frac{b}{a} = V_{s2}. \]

By reintroducing the time constant \( \tau = RC = 1/a \) and replacing \( b/a \) with \( v(\infty) \), we can rewrite Eq. (5.94) in the general form:

\[ v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \quad (\text{for } t \geq 0) \]

(switch action at \( t = 0 \)).

If the switch action causing the change in voltage across the capacitor occurs at time \( T_0 \) instead of at \( t = 0 \), Eq. (5.96) assumes the form

\[ v(t) = v(\infty) + [v(T_0) - v(\infty)]e^{-(t-T_0)/\tau} \quad (\text{for } t \geq T_0) \]

(switch action at \( t = T_0 \)).
Natural Response of the RL Circuit

\[ Ri + L \frac{di}{dt} = 0, \]

which can be cast in the form

\[ \frac{di}{dt} + ai = 0, \]

where \( a \) is a temporary constant given by

\[ a = \frac{R}{L}. \]

\[ i(t) = i(0) e^{-t/\tau} \]  
(for \( t \geq 0 \))  
(natural response),

where for the RL circuit, the \textit{time constant} is given by

\[ \tau = \frac{1}{a} = \frac{L}{R}. \]
General Response of the RL Circuit

\[-I_{s2} + i_R + i = 0.\]

Since \( v \) is common to \( R \) and \( L \), \( i_R = v/R \). and by applying \( v = L \frac{di}{dt} \), the KCL equation becomes

\[\frac{di}{dt} + ai = b,\]  

(5.105)

where \( a \) is as given previously by Eq. (5.102) and

\[b = aI_{s2} = \frac{R}{L} I_{s2}.\]  

(5.106)

\[i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \quad \text{(for } t \geq 0)\]  

\[\text{(switch action at } t = 0),\]

If the sudden change in the circuit configuration happens at \( t = T_0 \) instead of at \( t = 0 \), the general expression for \( i(t) \) becomes

\[i(t) = i(\infty) + [i(T_0) - i(\infty)]e^{-(t-T_0)/\tau} \quad \text{(for } t \geq T_0)\]  

\[\text{(switch action at } t = T_0),\]
RC Op-Amp Circuits: Ideal Integrator

\[ i_R = \frac{v_i}{R}. \]  \hspace{1cm} (5.123)

Given that \( v_n = 0 \), the voltage \( v_C \) across \( C \) is simply \( v_{\text{out}} \), and the current flowing through it is

\[ i_C = C \frac{dv_{\text{out}}}{dt}. \]  \hspace{1cm} (5.124)

At node \( v_n \),

\[ i_R + i_C - i_n = 0. \]  \hspace{1cm} (5.125)

In view of the second op-amp constraint, namely \( i_n = i_p = 0 \), it follows that

\[ i_C = -i_R \]  \hspace{1cm} (5.126)

or

\[ \frac{dv_{\text{out}}}{dt} = -\frac{1}{RC} v_i. \]  \hspace{1cm} (5.127)

Upon integrating both sides of Eq. (5.127) from a reference time \( t_0 \) to time \( t \), we have

\[ \int_{t_0}^{t} \left( \frac{dv_{\text{out}}}{dt} \right) dt = -\frac{1}{RC} \int_{t_0}^{t} v_i \, dt, \]  \hspace{1cm} (5.128)

which leads to

\[ v_{\text{out}}(t) = -\frac{1}{RC} \int_{t_0}^{t} v_i \, dt + v_{\text{out}}(t_0). \]  \hspace{1cm} (5.129)
RC Op-Amp Circuits: Ideal Differentiator

\[ i_C = C \frac{dv_i}{dt}, \]
\[ i_R = \frac{v_{out}}{R}, \]
\[ i_C = -i_R. \]

\[ v_{out} = -RC \frac{dv_i}{dt}, \]
Summary of Chapter 7

Mathematical and Physical Models

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<th>Trigonometric identities</th>
<th>Transformer</th>
<th>Wire resistance</th>
<th>Wire capacitor</th>
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<tr>
<td>Table 7-1</td>
<td>$\frac{v_2}{v_1} = \frac{N_2}{N_1}$</td>
<td>$R = \frac{2\ell}{\pi a^2 \sigma}$ for $(a\sqrt{f\sigma} \leq 500)$</td>
<td>$C = \frac{\pi \varepsilon \ell}{\ln(d/a)}$ for $(d/2a)^2 \gg 1$</td>
</tr>
<tr>
<td>Time domain/phasor domain correspondence</td>
<td>$\frac{i_2}{i_1} = \frac{N_1}{N_2}$</td>
<td>$R = \sqrt{\frac{\pi f \mu}{\sigma}} \left( \frac{\ell}{\pi a} \right)$ for $(a\sqrt{f\sigma} \geq 1250)$</td>
<td></td>
</tr>
<tr>
<td>Impedance</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_R = R$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_C = \frac{1}{j\omega C}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_L = j\omega L$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Impedances in series</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_{eq} = \sum_{i=1}^{N} Z_i$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Admittances in parallel</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Y_{eq} = \sum_{i=1}^{N} Y_i$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y–Δ transformation</td>
<td>Section 7-4.2</td>
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ac Analysis

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Objectives

Learn to:

- Transform time-varying sinusoidal functions to the phasor domain and vice versa.
- Analyze any linear circuit in the phasor domain.
- Determine the impedance of any passive element, or the combination of elements connected in series or in parallel.
- Perform source transformations, current division and voltage division, and determine Thévenin and Norton equivalent circuits, all in the phasor domain.

Electric circuits whose currents and voltages vary sinusoidally with time—called alternating current (ac) circuits—are at the heart of most analog applications. This chapter and the next four are dedicated to ac circuits.

- Apply nodal analysis, mesh analysis, and other analysis techniques, all in the phasor domain.
- Design simple RC phase-shift circuits.
- Design a dc power-supply circuit.
- Use Multisim to analyze ac circuits
Phasor Domain

A domain transformation is a mathematical process that converts a set of variables from their domain into a corresponding set of variables defined in another domain.

1. The phasor-analysis technique transforms equations from the time domain to the phasor domain.

2. Integro-differential equations get converted into linear equations with no sinusoidal functions.

3. After solving for the desired variable--such as a particular voltage or current--in the phasor domain, conversion back to the time domain provides the same solution that would have been obtained had the original integro-differential equations been solved entirely in the time domain.
# Phasor Domain

\[ v(t) = V_0 \cos(\omega t + \phi) \]
\[ = \Re[V_0 e^{j\phi} e^{j\omega t}] \]

Phasor counterpart of \( v(t) \)

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Phasor Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(t) = V_0 \cos \omega t )</td>
<td>( V = V_0 )</td>
</tr>
<tr>
<td>( v(t) = V_0 \cos(\omega t + \phi) )</td>
<td>( V = V_0 e^{j\phi} ).</td>
</tr>
</tbody>
</table>

If \( \phi = -\pi / 2 \),

\[ v(t) = V_0 \cos(\omega t - \pi / 2) \quad \leftrightarrow \quad V = V_0 e^{-j\pi / 2} \]
It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain.

You just need to track magnitude/phase, knowing that everything is at frequency $\omega$. 

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cos \omega t$</td>
<td>$A$</td>
</tr>
<tr>
<td>$A \cos(\omega t + \phi)$</td>
<td>$Ae^{j\phi}$</td>
</tr>
<tr>
<td>$-A \cos(\omega t + \phi)$</td>
<td>$Ae^{j(\phi+\pi)}$</td>
</tr>
<tr>
<td>$A \sin \omega t$</td>
<td>$Ae^{-j\pi/2} = -jA$</td>
</tr>
<tr>
<td>$A \sin(\omega t + \phi)$</td>
<td>$Ae^{j(\phi-\pi/2)}$</td>
</tr>
<tr>
<td>$-A \sin(\omega t + \phi)$</td>
<td>$Ae^{j(\phi+\pi/2)}$</td>
</tr>
<tr>
<td>$\frac{d}{dt}(x(t))$</td>
<td>$j\omega X$</td>
</tr>
<tr>
<td>$\frac{d}{dt}[A \cos(\omega t + \phi)]$</td>
<td>$j\omega Ae^{j\phi}$</td>
</tr>
<tr>
<td>$\int x(t) , dt$</td>
<td>$\frac{1}{j\omega} X$</td>
</tr>
<tr>
<td>$\int A \cos(\omega t + \phi) , dt$</td>
<td>$\frac{1}{j\omega} Ae^{j\phi}$</td>
</tr>
</tbody>
</table>
Phasor Relation for Resistors

Current through a resistor

**Time Domain**

\[ i = I_m \cos(\omega t + \phi) \]

\[ v = iR = R I_m \cos(\omega t + \phi) \]

**Frequency Domain**

\[ I \]

\[ V = R I \]

\[ V = RI_m \angle \phi \]

\[ v = iR \]
Phasor Relation for Inductors

Current through inductor in time domain
\[ i = I_m \cos(\omega t + \phi) \]

Time domain
\[ v = L \frac{di}{dt} \]

Phasor Domain
\[ v_L = \Re[V_L e^{j\omega t}] \]

and
\[ i_L = \Re[I_L e^{j\omega t}] \]

Consequently,
\[ \Re[V_L e^{j\omega t}] = L \frac{d}{dt} [\Re[I_L e^{j\omega t}]] \]
\[ = \Re[j\omega L I_L e^{j\omega t}], \]

which leads to
\[ V_L = j\omega L I_L \]

and
\[ Z_L = \frac{V_L}{I_L} = j\omega L. \]
Phasor Relation for Capacitors

Voltage across capacitor in time domain is

\[ v = V_m \cos(\omega t + \phi) \]

Time domain

\[ i = C \frac{dv}{dt} \]

Frequency Domain

\[ I = j\omega CV \]

Phasor Domain

\[ I_C = j\omega CV_C \]

\[ Z_C = \frac{V_C}{I_C} = \frac{1}{j\omega C}. \]
**Summary of R, L, C**

<table>
<thead>
<tr>
<th>Property</th>
<th>R</th>
<th>L</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v-i$</td>
<td>$v = Ri$</td>
<td>$v = L \frac{di}{dt}$</td>
<td>$i = C \frac{dv}{dt}$</td>
</tr>
<tr>
<td>V–I</td>
<td>$V = RI$</td>
<td>$V = j\omega LI$</td>
<td>$V = \frac{I}{j\omega C}$</td>
</tr>
<tr>
<td>$Z$</td>
<td>$R$</td>
<td>$j\omega L$</td>
<td>$\frac{1}{j\omega C}$</td>
</tr>
<tr>
<td>dc equivalent</td>
<td>R</td>
<td>Short circuit</td>
<td>Open circuit</td>
</tr>
<tr>
<td>High-frequency equivalent</td>
<td>R</td>
<td>Open circuit</td>
<td>Short circuit</td>
</tr>
</tbody>
</table>

**Frequency response**

- $|Z_R|$: $R$
- $|Z_L|$: $\omega L$
- $|Z_C|$: $\frac{1}{\omega C}$
ac Phasor Analysis General Procedure

Using this procedure, we can apply our techniques from dc analysis.
**Impedance and Admittance**

**Impedance is**
- voltage/current

$$Z = R + jX$$

- $R = \text{resistance} = \text{Re}(Z)$
- $X = \text{reactance} = \text{Im}(Z)$

**Admittance is**
- current/voltage

$$Y = \frac{1}{Z} = G + jB$$

- $G = \text{conductance} = \text{Re}(Y)$
- $B = \text{susceptance} = \text{Im}(Y)$

<table>
<thead>
<tr>
<th>Component</th>
<th>Impedance $Z$</th>
<th>Admittance $Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$Z = R$</td>
<td>$Y = \frac{1}{R}$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$Z = j\omega L$</td>
<td>$Y = \frac{1}{j\omega L}$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$Z = \frac{1}{j\omega C}$</td>
<td>$Y = j\omega C$</td>
</tr>
</tbody>
</table>
Impedance Transformation

(a) RL

\[ Z_1 = R_1 + j\omega L_1 \]

(b) RC

\[ Z_2 = R_2 - \frac{j}{\omega C_2} \]

(c) LC

\[ Z_3 = j\left(\omega L_3 - \frac{1}{\omega C_3}\right) \]
Voltage & Current Division

Voltage Division

\[ V_1 = \left( \frac{Z_1}{Z_1 + Z_2} \right) V_s \]

\[ V_2 = \left( \frac{Z_2}{Z_1 + Z_2} \right) V_s \]

Current Division

\[ I_1 = \left( \frac{Y_1}{Y_1 + Y_2} \right) I_s \]

\[ I_2 = \left( \frac{Y_2}{Y_1 + Y_2} \right) I_s \]
Linear Circuit Properties

Thévenin/Norton and Source Transformation Also Valid

(a) $v_s(t) = 10 \cos 10^5 t$ (V)

(f) Thévenin equivalent

$V_{Th(t)} = 7.6 \cos (10^5 t - 31.61^\circ)$ V

$R_{Th} = 8.42 \Omega$

$C_{Th} = 6.29 \mu F$