LECTURE 9 RC AND RL FIRST-ORDER CIRCUITS (PART 2)

RC and RL First-Order Circuits

Table 5-4: Basic properties of R, L, and C.

<table>
<thead>
<tr>
<th>Property</th>
<th>R</th>
<th>L</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = \frac{v}{R} )</td>
<td>( i = \frac{1}{L} \int dt + i(t) )</td>
<td>( i = \frac{v}{C} )</td>
<td></td>
</tr>
<tr>
<td>( v = Ri )</td>
<td>( v = L \frac{di}{dt} )</td>
<td>( v = C \frac{di}{dt} )</td>
<td></td>
</tr>
<tr>
<td>( P ) (power transfer)</td>
<td>( P = \frac{1}{2} R i^2 )</td>
<td>( P = \frac{1}{2} L \frac{di^2}{dt} )</td>
<td></td>
</tr>
<tr>
<td>( U ) (internal energy)</td>
<td>0</td>
<td>( U = \frac{1}{2} L i^2 )</td>
<td>( U = \frac{1}{2} C v^2 )</td>
</tr>
<tr>
<td>Series combination</td>
<td>( R_{eq} = R_1 + R_2 )</td>
<td>( L_{eq} = L_1 + L_2 )</td>
<td>( C_{eq} = \frac{C_1}{C_1 + C_2} )</td>
</tr>
<tr>
<td>Parallel combination</td>
<td>( R_{eq} = \frac{R_1 R_2}{R_1 + R_2} )</td>
<td>( L_{eq} = \frac{L_1 L_2}{L_1 + L_2} )</td>
<td>( C_{eq} = C_1 + C_2 )</td>
</tr>
<tr>
<td>( \Delta ) behavior</td>
<td>no-change</td>
<td>short circuit</td>
<td>open-circuit</td>
</tr>
<tr>
<td>Can ( v ) change instantaneously?</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Can ( i ) change instantaneously?</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>
Response Terminology

Source dependence
Natural response – response in absence of sources
Forced response – response due to external source
Complete response = Natural + Forced

Time dependence
Transient response = time-varying response (temporary)
Steady state response = time-independent or periodic (permanent)
Complete response = Transient + Steady State

Natural Response of Charged Capacitor

\[ t = 0^- \] is the instant just before the switch is moved from terminal 1 to terminal 2
\[ t = 0^+ \] is the instant just after it was moved
\[ t = 0^+ \] is synonymous with \[ t = 0^+ \]
since the voltage across the capacitor cannot change instantaneously. It follows that

\[ \text{Solution of First-Order Diff. Equations} \]

The standard procedure for solving Eq. (5.5) starts by multiplying both sides by \( e^{\frac{t}{R}} \).

Performing the integration gives

\[ \frac{d}{dt} \left( e^{\frac{t}{R}} \right) = 0, \quad (5.52) \]

Next, we recognize that the sum of the two terms on the left-hand side is equal to the expression of the differential of \( e^{\frac{t}{R}} \).

Solving for \( \psi \), we have

\[ \int e^{\frac{t}{R}} \, dt = \int 0 \, dt, \quad (5.53) \]

Hence, Eq. (5.52) becomes

\[ \frac{d}{dt} \left( e^{\frac{t}{R}} \right) = 0; \quad (5.54) \]

Integrating both sides, we have

\[ \int e^{\frac{t}{R}} \, dt = \int 0 \, dt, \quad (5.55) \]

\( t = RC \) (6)
Natural Response of Charged Capacitor

$$i(t) = C \frac{dV}{dt} = C \frac{d}{dt}(V_0 e^{-t/\tau})$$
$$= -\frac{V_0}{\tau} e^{-t/\tau} \quad \text{(for } t \geq 0)$$

which simplifies to

$$i(t) = \frac{V_0}{R} e^{-t/\tau} u(t) \quad \text{(for } t \geq 0)$$

where \( u(t) \) is the unit step function.

The current \( i(t) \) decays exponentially to zero as \( t \) increases.

General Response of RC Circuit

The voltage equation for the loop in Fig. 5.5(c) is

$$v_i(t) + i(t) R + v(t) = 0$$

Using \( i = C \frac{dv}{dt} \) and rearranging its terms, Eq. (5.18) can be written in the differential equation form

$$\frac{dv}{dt} + \frac{1}{RC} v = 0$$

where \( \alpha = \frac{1}{RC} \) and \( \beta = \frac{V_0}{RC} \).

Solution of

By introducing the time constant \( \tau = RC \) and replacing \( \beta \) with \( v(t) \), we can rewrite Eq. (5.18) in the general form.

$$v(t) = \alpha (1 + e^{-\alpha t})$$

Upon evaluating the function at the two limits, we have

$$v(0) = \alpha (1 + e^{-\alpha \cdot 0}) = \beta$$

and upon solving for \( \beta \), we have

$$\beta = \alpha (1 + e^{-\alpha \cdot 0})$$

And upon solving for \( v(t) \), we have

$$v(t) = \beta (1 - e^{-\alpha t})$$

For the switch action causing the change in voltage across the capacitor occurs at time \( t = T_0 \) instead of \( t = 0 \), Eq. (5.18) assumes the form

$$v(t) = \beta (1 - e^{-\alpha (t - T_0)})$$

switch closure at \( t = T_0 \).
Example 5-10: Determine Capacitor Voltage

At \( t = 0 \)
(a) Switch was moved at \( t = 0 \)

\[ v(t) = v(0) + \frac{1}{C} \int i(t) \, dt \]

\[ v(t) = \left( \frac{16}{3} \right) + \left( \frac{1}{10^3} \right) \times 30 = 30 \text{ V} \]

Hence,
\[ R = \frac{10^3}{2} = 5 \times 10^3 \text{ k} \Omega \]

(b) Switch was moved at \( t > 0 \)

At \( t > 0 \)

Example 5-11: Charge/Discharge Action

Given that the switch in Fig. 5.32 was moved to position 2 at \( t = 0 \) and then returned to position 1 at \( t = 10 \text{ s} \), determine the voltage across \( C \) for \( t > 0 \) and calculate it for \( t = 10 \text{ s} \).

\( R_1 = 20 \text{ k} \Omega \), \( R_2 = 30 \text{ k} \Omega \), and \( C = 0.25 \mu \text{F} \).

Time Segment I: \( 0 \leq t \leq 10 \text{ s} \)

When the switch is in position 2 (Fig. 5.32(b)), the resistance of the circuit is \( R = R_2 \). Hence, the time constant during this time segment is:

\[ \tau = R \times C = \left( 30 \times 10^3 \right) \times 0.25 \times 10^{-6} = 7.5 \text{ s} \]

\[ v(t) = v(0) + \left( \frac{v(t)}{v(t)} \right) \int t \tau \, dt = v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt \]

\[ v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt = v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt \]

\[ v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt = v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt \]

\[ v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt = v(t) = v(0) + \left( \frac{1}{v(t)} \right) \times \int t \tau \, dt \]
Example 5-11 (cont.)

Voltage \( v(t) \), corresponding to the second time segment (Fig. 5-2(c)), is given by Eq. 5.30 with time constant \( t_0 \) as:

\[ v(t) = v(t_0) e^{-t/t_0}. \]

The time constant is associated with the equivalent circuit remaining after turning the switch to position 1.

\[ t_0 = RC = 20 \times 10^3 \times 0.25 \times 10^{-3} = 5 \text{ s}. \]

The initial voltage \( v(t) = 12 \text{ V} \) is equal to the capacitor voltage \( v_C \) at the end of the time segment 1, namely:

\[ v(t_0) = v_C = 12 V = 6.5 \text{ V} \text{ at } t = 10 \text{ s}. \]

Example 5-12: Rectangular Pulse

When the circuit is linear, we can apply the superposition theorem to determine the response \( v(t) \). Thus,

\[ \begin{align*}
    \text{(a)} & \quad k(t) = \frac{v(t)}{R} = \frac{12}{10} = 1.2 \text{ V} \\
    \text{(b)} & \quad k(t) = \frac{v(t)}{R} = \frac{12}{10} = 1.2 \text{ V} \\
    \text{(c)} & \quad k(t) = \frac{v(t)}{R} = \frac{12}{10} = 1.2 \text{ V} \\
    \text{(d)} & \quad k(t) = \frac{v(t)}{R} = \frac{12}{10} = 1.2 \text{ V} \\
    \text{(e)} & \quad k(t) = \frac{v(t)}{R} = \frac{12}{10} = 1.2 \text{ V}.
\end{align*} \]

Natural Response of the RL Circuit

\[ R_0 + \frac{L}{C} \frac{dI}{dt} = 0, \]

which can be cast in the form

\[ \frac{d}{dt} \left( \frac{1}{2} \right) = \frac{1}{L} \]

where \( R_0 \) is the resistance constant given by

\[ r = \frac{1}{L}. \]

where for the RL circuit, the time constant is given by

\[ t = \frac{L}{R}. \]
Example 5-13: Two RL Branches

After having been in position 1 for a long time, the SPDT switch in Fig. 5-13(a) was moved to position 2 at \(t = 0\). Determine \(i_1\), \(i_2\), and \(i_3\) for \(t \geq 0\), given that \(V_s = 9.0\,\text{V}\), \(R_s = 4.0\,\Omega\), \(R_1 = 6.0\,\text{k}\Omega\), \(R_2 = 12.0\,\Omega\), \(L_1 = 1.2\,\text{H}\), and \(L_2 = 0.86\,\text{H}\).

At \(t = 0^+\):

Application of KVL to node \(V\) gives:

\[
\frac{V_s}{R_s} = \frac{V}{R_1} + \frac{V}{R_2} = 0.
\]

where solution is:

\[
\frac{V_s}{R_s} = \frac{R_2}{R_1 + R_2 + R_s} = \frac{12.0}{12.0 + 6.0} = 0.68\,\text{V}.
\]

Hence, the initial currents \(i(t_0)\) and \(i'(t_0)\) are given by:

\[
i(t_0) = \frac{V}{R_1} = \frac{4.8}{12.0} = 0.4\,\text{mA}
\]

and

\[
i'(t_0) = \frac{V}{R_2} = \frac{4.8}{6.0} = 0.8\,\text{mA}.
\]

Example 5-13: Two RL Branches (cont.)

After \(t_0\):

\[
i(t) = i(t_0) + i_0(t) = i(t_0) + \frac{i}{i} + \left[\left(\frac{1}{\tau_1}\right)\,e^{-t/\tau_1}\right].
\]

and

\[
i(t) = i(t_0) + i_0(t) = i(t_0) + \frac{i}{i} + \left[\left(\frac{1}{\tau_2}\right)\,e^{-t/\tau_2}\right],
\]

where \(i_0\) and \(i_1\) are the time constants of the two RL circuit branches

\[
\tau_1 = \frac{L_1}{R_1} = \frac{1.2}{6.0} = 2.0 \times 10^{-6}\,\text{s}
\]

and

\[
\tau_2 = \frac{L_2}{R_2} = \frac{0.36}{12.0} = 3.0 \times 10^{-6}\,\text{s}.
\]

The current flowing through the short circuit is simply:

\[
i(t) = i_0(t) + \frac{i}{i} + \left[\left(\frac{1}{\tau_1}\right)\,e^{-t/\tau_1}\right] + \left[\left(\frac{1}{\tau_2}\right)\,e^{-t/\tau_2}\right].
\]