8.1. Review of Bode plots

**Decibels**

\[ \|G\|_{dB} = 20 \log_{10}\left(\|G\|\right) \]

**Table 8.1. Expressing magnitudes in decibels**

<table>
<thead>
<tr>
<th>Actual magnitude</th>
<th>Magnitude in dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>-6 dB</td>
</tr>
<tr>
<td>1</td>
<td>0 dB</td>
</tr>
<tr>
<td>2</td>
<td>6 dB</td>
</tr>
<tr>
<td>5 = 10/2</td>
<td>20 dB - 6 dB = 14 dB</td>
</tr>
<tr>
<td>10</td>
<td>20 dB</td>
</tr>
<tr>
<td>1000 = 10^3</td>
<td>3 \cdot 20 dB = 60 dB</td>
</tr>
</tbody>
</table>

5\(\Omega\) is equivalent to 14dB with respect to a base impedance of \(R_{base} = 1\Omega\), also known as 14dB\(\Omega\).

60dB\(\mu\)A is a current 60dB greater than a base current of 1\(\mu\)A, or 1mA.
Bode plot of $f^n$

Bode plots are effectively log-log plots, which cause functions which vary as $f^n$ to become linear plots. Given:

$$\|G\| = \left(\frac{f}{f_0}\right)^n$$

Magnitude in dB is

$$\|G\|_{\text{dB}} = 20 \log_{10} \left(\frac{f}{f_0}\right)^n = 20n \log_{10} \left(\frac{f}{f_0}\right)$$

- Slope is $20n$ dB/decade
- Magnitude is 1, or 0dB, at frequency $f = f_0$
8.1.1. Single pole response

Simple R-C example

Transfer function is

\[ G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{sC} + R \]

Express as rational fraction:

\[ G(s) = \frac{1}{1 + sRC} \]

This coincides with the normalized form

\[ G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)} \]

with \( \omega_0 = \frac{1}{RC} \)
Let $s = j\omega$:

$$G(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Magnitude is

$$\|G(j\omega)\| = \sqrt{\left[\text{Re} \ (G(j\omega))\right]^2 + \left[\text{Im} \ (G(j\omega))\right]^2}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

Magnitude in dB:

$$\|G(j\omega)\|_{db} = -20 \log_{10} \left(\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}\right) \text{ dB}$$
For small frequency, \( \omega \ll \omega_0 \) and \( f \ll f_0 \):

\[
\left( \frac{\omega}{\omega_0} \right) \ll 1
\]

Then \( \| G(j\omega) \| \) becomes

\[
\| G(j\omega) \| \approx \frac{1}{\sqrt{1}} = 1
\]

Or, in dB,

\[
\| G(j\omega) \|_{dB} \approx 0 \text{dB}
\]

This is the low-frequency asymptote of \( \| G(j\omega) \| \)
Asymptotic behavior: high frequency

For high frequency, \( \omega >> \omega_0 \) and \( f >> f_0 \):

\[
\left( \frac{\omega}{\omega_0} \right) >> 1
\]

\[
1 + \left( \frac{\omega}{\omega_0} \right)^2 \approx \left( \frac{\omega}{\omega_0} \right)^2
\]

Then \( \| G(j\omega) \| \) becomes

\[
\| G(j\omega) \| \approx \frac{1}{\sqrt{\left( \frac{\omega}{\omega_0} \right)^2}} = \left( \frac{f}{f_0} \right)^{-1}
\]

The high-frequency asymptote of \( \| G(j\omega) \| \) varies as \( f^{-1} \). Hence, \( n = -1 \), and a straight-line asymptote having a slope of -20dB/decade is obtained. The asymptote has a value of 1 at \( f = f_0 \).
Deviation of exact curve near $f = f_0$

Evaluate exact magnitude:

at $f = f_0$:

$$
\left| G(j\omega_0) \right| = \frac{1}{\sqrt{1 + \left( \frac{\omega_0}{\omega_0} \right)^2}} = \frac{1}{\sqrt{2}}
$$

$$
\left| G(j\omega_0) \right|_{\text{dB}} = -20 \log_{10} \left( \sqrt{1 + \left( \frac{\omega_0}{\omega_0} \right)^2} \right) \approx -3 \text{ dB}
$$

at $f = 0.5f_0$ and $2f_0$:

Similar arguments show that the exact curve lies 1dB below the asymptotes.
Summary: magnitude

$$\| G(j\omega) \|_{dB}$$

- $0$ dB
- $1$ dB
- $0.5f_0$
- $f_0$
- $2f_0$
- $3$ dB
- $-20$ dB/decade
- $-20$ dB
- $-30$ dB

$f$
Phase of $G(j\omega)$

$$G(j\omega) = \frac{1}{1 + j \frac{\omega}{\omega_0}} = \frac{1 - j \frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))}\right)$$
Phase of $G(j\omega)$

\[ \angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \]

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\angle G(j\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0°</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>-45°</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-90°</td>
</tr>
</tbody>
</table>
Phase asymptotes

Low frequency: 0°
High frequency: –90°
Low- and high-frequency asymptotes do not intersect
Hence, need a midfrequency asymptote

Try a midfrequency asymptote having slope identical to actual slope at the corner frequency $f_0$. One can show that the asymptotes then intersect at the break frequencies

\[
f_a = f_0 e^{-\frac{\pi}{2}} \approx \frac{f_0}{4.81}
\]
\[
f_b = f_0 e^{\frac{\pi}{2}} \approx 4.81 f_0
\]
Phase asymptotes

\[ f_a = f_0 e^{-\pi/2} \approx f_0 / 4.81 \]
\[ f_b = f_0 e^{\pi/2} \approx 4.81 f_0 \]
Phase asymptotes: a simpler choice

\[ f_a = f_0 / 10 \]
\[ f_b = 10 f_0 \]
Summary: Bode plot of real pole

\[ G(s) = \frac{1}{1 + \frac{s}{\omega_0}} \]
8.1.2. Single zero response

Normalized form:

\[ G(s) = \left( 1 + \frac{s}{\omega_0} \right) \]

Magnitude:

\[ \| G(j\omega) \| = \sqrt{1 + \left( \frac{\omega}{\omega_0} \right)^2} \]

Use arguments similar to those used for the simple pole, to derive asymptotes:

- 0dB at low frequency, \( \omega << \omega_0 \)
- +20dB/decade slope at high frequency, \( \omega >> \omega_0 \)

Phase:

\[ \angle G(j\omega) = \tan^{-1} \left( \frac{\omega}{\omega_0} \right) \]

—with the exception of a missing minus sign, same as simple pole
Summary: Bode plot, real zero

\[ G(s) = \left( 1 + \frac{s}{\omega_0} \right) \]

\[ \| G(j\omega) \|_{dB} = 0 \text{dB} \quad 1 \text{dB} \quad 3 \text{dB} \]

\[ \angle G(j\omega) = 0^\circ \quad 5.7^\circ \quad 45^\circ \quad 90^\circ \]

+20dB/decade
+45°/decade
+90°
8.1.3. Right half-plane zero

Normalized form:

\[ G(s) = \left(1 - \frac{s}{\omega_0}\right) \]

Magnitude:

\[ \|G(j\omega)\| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \]

—same as conventional (left half-plane) zero. Hence, magnitude asymptotes are identical to those of LHP zero.

Phase:

\[ \angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \]

—same as real pole.

The RHP zero exhibits the magnitude asymptotes of the LHP zero, and the phase asymptotes of the pole.
Summary: Bode plot, RHP zero

\[ G(s) = \left( 1 - \frac{s}{\omega_0} \right) \]

\[ \| G(j\omega) \|_{\text{dB}} \]

\[ \angle G(j\omega) \]

-20dB/decade

-45°/decade

-90°
8.1.4. Frequency inversion

Reversal of frequency axis. A useful form when describing mid- or high-frequency flat asymptotes. Normalized form, inverted pole:

\[ G(s) = \frac{1}{1 + \frac{\omega_0}{s}} \]

An algebraically equivalent form:

\[ G(s) = \frac{\frac{s}{\omega_0}}{1 + \frac{s}{\omega_0}} \]

The inverted-pole format emphasizes the high-frequency gain.
Asymptotes, inverted pole

\[ G(s) = \frac{1}{1 + \omega_0 \frac{s}{s}} \]

\[ \| G(j\omega) \|_{dB} \]

\[ \angle G(j\omega) \]

\[ +20 \text{dB/decade} \]

\[ +90^\circ \]

\[ -45^\circ/\text{decade} \]

\[ +45^\circ \]

\[ 0^\circ \]

\[ 0 \text{dB} \]

\[ +3 \text{dB} \]

\[ 1 \text{dB} \]

\[ 0.5f_0 \]

\[ f_0 \]

\[ 2f_0 \]

\[ 10f_0 \]
Inverted zero

Normalized form, inverted zero:

\[ G(s) = \left( 1 + \frac{\omega_0}{s} \right) \]

An algebraically equivalent form:

\[ G(s) = \frac{1 + \frac{s}{\omega_0}}{\left( \frac{s}{\omega_0} \right)} \]

Again, the inverted-zero format emphasizes the high-frequency gain.
Asymptotes, inverted zero

\[ G(s) = \left( 1 + \frac{\omega_0}{s} \right) \]

\[ |G(j\omega)|_{\text{dB}} \]

\[ \angle G(j\omega) \]

+45°/decade

-90°

0°

-20dB/decade

0.5f_0

f_0

2f_0

1dB

3dB

1dB

10f_0

5.7°

5.7°

f_0 / 10
8.1.5. Combinations

Suppose that we have constructed the Bode diagrams of two complex-values functions of frequency, $G_1(\omega)$ and $G_2(\omega)$. It is desired to construct the Bode diagram of the product, $G_3(\omega) = G_1(\omega) G_2(\omega)$.

Express the complex-valued functions in polar form:

$$G_1(\omega) = R_1(\omega) \ e^{j\theta_1(\omega)}$$
$$G_2(\omega) = R_2(\omega) \ e^{j\theta_2(\omega)}$$
$$G_3(\omega) = R_3(\omega) \ e^{j\theta_3(\omega)}$$

The product $G_3(\omega)$ can then be written

$$G_3(\omega) = G_1(\omega) G_2(\omega) = R_1(\omega) \ e^{j\theta_1(\omega)} R_2(\omega) \ e^{j\theta_2(\omega)}$$

$$G_3(\omega) = \left( R_1(\omega) R_2(\omega) \right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$
Combinations

\[ G_3(\omega) = \left( R_1(\omega) R_2(\omega) \right) e^{j(\theta_1(\omega) + \theta_2(\omega))} \]

The composite phase is

\[ \theta_3(\omega) = \theta_1(\omega) + \theta_2(\omega) \]

The composite magnitude is

\[ R_3(\omega) = R_1(\omega) R_2(\omega) \]

\[ \left| R_3(\omega) \right|_{\text{dB}} = \left| R_1(\omega) \right|_{\text{dB}} + \left| R_2(\omega) \right|_{\text{dB}} \]

Composite phase is sum of individual phases.

Composite magnitude, when expressed in dB, is sum of individual magnitudes.
Example 1: \[ G(s) = \frac{G_0}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)} \]

with \( G_0 = 40 \Rightarrow 32 \text{ dB}, f_1 = \frac{\omega_1}{2\pi} = 100 \text{ Hz}, f_2 = \frac{\omega_2}{2\pi} = 2 \text{ kHz} \)
Example 2

Determine the transfer function $A(s)$ corresponding to the following asymptotes:

$||A||$

$||A_0||_dB$

$+20$ dB/dec

$\angle A$

$+45^\circ$/dec

$-90^\circ$

$-45^\circ$/dec

$0^\circ$
Example 2, continued

One solution:

\[
A(s) = A_0 \frac{(1 + \frac{s}{\omega_1})}{(1 + \frac{s}{\omega_2})}
\]

Analytical expressions for asymptotes:

For \( f < f_1 \)

\[
\left. A_0 \left(1 + \frac{s}{\omega_1}\right) \right|_{s = j\omega} = A_0 \frac{1}{1} = A_0
\]

For \( f_1 < f < f_2 \)

\[
\left. A_0 \left(1 + \frac{s}{\omega_1}\right) \right|_{s = j\omega} = A_0 \frac{\omega_1}{1} = A_0 \frac{\omega}{\omega_1} = A_0 \frac{f}{f_1}
\]