Theoretical Foundations of Spatially-Variant Mathematical Morphology Part II: Gray-Level Images

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Abstract—In this paper, we develop a spatially-variant (SV) mathematical morphology theory for gray-level signals and images in the Euclidean space. The proposed theory preserves the geometrical concept of the structuring function, which provides the foundation of classical morphology and is essential in signal and image processing applications. We define the basic SV gray-level morphological operators (i.e., SV gray-level erosion, dilation, opening, and closing) and investigate their properties. We demonstrate the ubiquity of SV gray-level morphological systems by deriving a kernel representation for a large class of systems, called V-systems, in terms of the basic SV gray-level morphological operators. A V-system is defined to be a gray-level operator, which is invariant under gray-level (vertical) translations. Particular attention is focused on the class of SV flat gray-level operators. The kernel representation for increasing V-systems is a generalization of Maragos’ kernel representation for increasing and translation-invariant function-processing systems. A representation of V-systems in terms of their kernel elements is established for increasing and upper-semi-continuous V-systems. This representation unionizes a large class of spatially-variant linear and non-linear systems under the same mathematical framework. The theory is used to analyze special cases of signal and image processing systems such as SV order rank filters and linear-time-varying systems. Finally, simulation results show the potential power of the general theory of gray-level spatially-variant mathematical morphology in several image analysis and computer vision applications.

Index Terms—spatially-variant mathematical morphology, gray-level morphology, upper-semi-continuous functions, adaptive order-statistic filters, linear-time-varying systems.

I. INTRODUCTION

Originally, mathematical morphology theory was developed for translation-invariant transformations of binary images (or two-level), i.e., operators which are invariant under the Euclidean group of translations [1], [2]. The theory has subsequently been extended to translation-invariant transformations of gray-level (or multi-level) images by Sternberg [3], Serra [2] and Maragos [4], [5]. A translation-invariant gray-level transformation is defined to be invariant under horizontal (space or time in 1-D) translations and vertical (gray-level or signal values) translations. In mathematical morphology, sets are used as mathematical representations of binary signals and images; whereas functions represent gray-level signals. This characterization induces a similar classification for systems 1 into either function-processing (FP) systems, which accept as inputs and produce as outputs multilevel signals, or set-processing (SP) systems, whose inputs and outputs are binary signals [5]. The extension of translation-invariant binary morphology to the gray-level case was first derived based on set representation of functions. There are two different but equivalent approaches to represent a function by a set or an equivalent class of sets: The umbra approach and the threshold sets method. The umbra approach, which was introduced by Serra [2], relies on the fact that the points on and below the graph of a function correspond to a set representation of the function in a higher-dimensional space. The threshold sets method, introduced by Serra [2], represents a function by an equivalent class of sets, called cross-sections or threshold sets, by thresholding it at successive levels. Although the umbra approach has a nice geometrical interpretation of the morphological gray-level operations in terms of their binary counterparts, it may be the source of many mistakes if not handled properly [6], [7]. The problem with the umbra approach is that, in general, the union of a collection of umbras is not an umbra, and consequently the binary dilation of two umbras is not necessarily an umbra again [7]. However, these technical difficulties vanish if we restrict the space to the set of umbras of upper-semi-continuous (u.s.c.) functions 2. Both Serra [8] and Maragos and Schafer [9] restrict their presentation of the extension of set operators to function operators to u.s.c. functions. This restriction is not necessary if one wants to construct gray-level operators from set operators by thresholding. However, the threshold sets method is limited to systems, which commute with thresholding (e.g., flat structuring functions).

The extension of Matheron’s kernel representation theorem for translation-invariant set-processing systems [1] to function-processing systems was carried out by Maragos [4], [5]. He showed that every increasing and translation-invariant function-processing system can be represented as supremum (resp. infimum) of function-processing erosions (resp. dilations). Furthermore, Maragos showed that the kernel representation of function-processing systems is redundant, in the

1In this paper, we will use interchangeably “operator” and “system” to denote processes that accept as inputs and produce as outputs multidimensional signals.

2Upper-semi-continuity is a property of functions that is weaker than continuity. If is upper-semi-continuous if for all \( x_0 \in E \), \( \lim \sup_{x \to x_0} f(x) \leq f(x_0) \).
sense that a smaller subset of the kernel is sufficient for the representation of the system. Subsequently, he provided sufficient conditions for translation-invariant function-processing systems to admit a basis representation [4], [5]. Heijmans [7], [10] relaxed the translation-invariance assumption by studying function-processing systems which are invariant under horizontal translations, the so-called H-operators. However, so far, no comprehensive mathematical framework has been presented to establish the foundations of spatially-variant (SV) gray-level mathematical morphology in the Euclidean space.

Following Serra’s work in [2, Chaps. 2,4], we elaborated in [11] on the general theory of spatially-variant mathematical morphology in the Euclidean space for binary signals. This theory captures the geometrical interpretation of the structuring element (SE), which is crucial in signal and image processing applications. This paper extends the theory of spatially-variant mathematical morphology presented in [11] to the gray-level case. Specifically, we consider the class of V-systems, which are function-processing systems that are spatially-variant and invariant under gray-level translations. In other words, the structuring function varies in space independently of the signal values. V-systems have been used extensively in adaptive filtering applications [12], [13], [14], [15], [16], [17], [18], [19], [20]. Moreover, morphological V-systems have an elegant geometric interpretation which is consistent with their translation-invariant counterparts. In this paper, we define the basic spatially-variant function-processing (SVFP) morphological operators (i.e., SVFP erosion, dilation, opening, and closing) and investigate their properties. We show that the basic properties of translation-invariant function-processing morphological systems [21] can be transposed to SVFP systems. Special focus is devoted to the class of V-systems which commute with thresholding. The class of translation-invariant systems, which commute with thresholding, has been extensively studied in the literature, under many different names. Heijmans [7] calls them flat operators; Maragos [4], [5] refers to them as function-set-processing systems and Wendt et al. [22] denote them by stack filters. In our presentation, we will refer to V-systems, which commute with thresholding, as spatially-variant function-set-processing (SVFSP) systems, as this nomenclature is more appealing to the signal and image processing community. We demonstrate the ubiquity of the basic SVFP and SVFSP morphological systems by providing a SV kernel representation for V-systems. Specifically, we prove that every increasing V-system can be represented as supremum (resp. infimum) of SVFP erosions (resp. SVFP dilations). The latter kernel representation is a generalization of Maragos’ kernel representation for increasing and translation-invariant function-processing systems [4], [5]. Furthermore, based on Maragos’ development of the basis representation for translation-invariant function-processing systems [4], [5], we provide sufficient conditions for the existence of a basis representation for V-systems. Examples, provided throughout the paper, demonstrate that the proposed SV gray-level mathematical morphology unifies different methods in adaptive gray-level morphology such as adaptive neighborhood morphology [18], [19] and vertically-invariant morphology [17]. Our goal is to provide a sound mathematical framework to unify current and future research in spatially-variant morphological signal processing and to provide the mathematical tools needed for the design and analysis of spatially-variant morphological filters in image analysis and computer vision applications.

This paper is organized as follows: In Section II, we define the basic SVFP and SVFSP morphological systems. Their properties are investigated in Appendix A. In Section III, we establish a kernel representation for increasing V-systems and a basis representation for increasing and upper-semi-continuous V-systems. Section IV illustrates the theory through the study of two adaptive systems: SV order statistic filters and linear time-varying systems. Simulation results, in Section V, show the power of the proposed theory of spatially-variant gray-level morphology in denoising, multiscale filtering and segmentation. Finally, a summary of the results of this paper is provided in Section VI.

The proofs of all theoretical results that are new contributions in this paper have been inserted in the appendix.

II. SPATIALLY-VARIANT FUNCTION-PROCESSING BASIC MORPHOLOGICAL SYSTEMS

A. Preliminaries

In this paper, we consider the space $\mathbb{E} = \mathbb{R}^m$ or $\mathbb{Z}^m$ for some $m \geq 1$. The power set of $\mathbb{E}$ will be denoted by $\mathcal{P}(\mathbb{E})$. A gray-level signal is a function from $\mathbb{E}$ to a gray-level space $\mathcal{T}$, where $\mathcal{T} = \mathbb{R}$ or $\mathbb{Z}$. The collection of such functions is denoted as $Func(\mathbb{E})$. The least and greatest elements of $Func(\mathbb{E})$ are denoted by $\mathcal{O}$ and $\mathcal{I}$; these are the functions identically equal to $-\infty$ and $+\infty$, respectively. An important subset of $Func(\mathbb{E})$ is the collection of upper-semi-continuous (u.s.c.) functions [23], [9] denoted by $USC(\mathbb{E})$. In this paper, we will only consider u.s.c. functions. Elements of $USC(\mathbb{E})$ will be denoted by lower case letters; e.g., $f, g$. Set-processing systems will be denoted by lower-case Greek letters; e.g., $\psi, \phi$; whereas function-processing systems will be denoted by upper case Greek letters; e.g., $\Psi, \Phi$. “$\iff”, “\leftrightarrow, \forall, \exists$” denote, respectively, “implies,” “if and only if,” “for all,” and “there exist(s).” The support of a function $f$ is defined as $Spt(f) = \{ x \in \mathbb{E} : f(x) \neq -\infty \}$. The umbra, $U[f]$, of a function $f$ is defined by

$$U[f] = \{(x, y) \in \mathbb{E} \times \mathcal{T} : y \leq f(x)\}. \tag{1}$$

The threshold set of the function $f$ at level $t$ is given by

$$\mathcal{X}_t(f) = \{ x \in \mathbb{E} : f(x) \geq t \}. \tag{2}$$

The reflected function $\tilde{f}$ of a function $f$ is defined as $\tilde{f}(x) = f(-x), \forall x \in \mathbb{E}$. The horizontal (or spatial) translate, $f_a$, of a function $f$ by $a \in \mathbb{E}$ is defined as $f_a(x) = f(x-a), \forall x \in \mathbb{E}$. The vertical translate, $f + b$, of the function $f$ by $b \in \mathcal{T}$, is defined by $(f + b)(x) = f(x) + b, \forall x \in \mathbb{E}$. The translation of a function $f$ by the vector $(a, b) \in \mathbb{E} \times \mathcal{T}$ is defined as $f_{(a,b)}(x) = f(x-a) + b, \forall x \in \mathbb{E}$. The counterpart of the set complementation for functions is the function negation, defined by $f^*(x) = -f(x), \forall x \in \mathbb{E}$. An order is imposed on $USC(\mathbb{E})$ by setting $f \leq g$ iff $f(x) \leq g(x), \forall x \in \mathbb{E}$.
The latter order induces an order on the class of function-processing systems by setting $\Psi_1 \leq \Psi_2$ if $\Psi_1(f) \leq \Psi_2(f)$, $\forall f \in USC(E)$. $\wedge$ and $\vee$ denote the supremum and infimum operations, respectively. The counterpart of the dual operator for a function-processing system $\Psi$ is the negative function-processing system, $\Psi^\ast$, defined by $\Psi^\ast(f) = -\Psi(-f)$, $\forall f \in USC(E)$. In this paper, we consider only non-degenerate FP (resp., SP) systems, i.e., $\Psi(I) = I$ (resp., $\psi(E) = E$) and $\Psi(O) = O$ (resp., $\psi(0) = 0$).

**B. Spatially-Variant Function-Processing Morphological Systems**

The spatially-variant structuring function $\Theta$ is a mapping from $E$ to $USC(E)$, which associates to each point $x \in E$ an upper-semi-continuous structuring function $\Theta(x)$. The transposed structuring function mapping is given by

$$\Theta'(x)[u] = [\Theta(u)](x), \quad \forall x, u \in E. \quad (3)$$

In the translation-invariant case, the structuring function mapping is the horizontal translation operator of a fixed structuring function $g$, i.e., $\Theta(x) = g_x$, $\forall x \in E$. Then, $[\Theta'(x)[u] = [\Theta(u)](x) = g_x(u) = g(x - u) = g(u - x) = g_x(u)$, $\forall x, u \in E$. So, $\Theta'(x) = g_x$, $\forall x \in E$. That is, the transposed structuring function mapping reduces to the translation of the reflected function $g$. Therefore, the definition of the SV structuring function is consistent with the translation-invariant case. This analogy might give the impression that the structuring function mappings $\Theta$ and $\Theta'$ are the same up to a symmetry. This is not true in general: For example, consider the structuring function mapping that associates to each point in space a line through the origin with varying slope. Then, the transposed structuring function mapping assigns a hyperbola function at each point in space.

An order on the mappings from $E$ to $USC(E)$ is induced by the order on the space $USC(E)$; i.e., $\Theta_1 \leq \Theta_2$ if and only if $\Theta_1(x) \leq \Theta_2(x)$ for every $x \in E$. We say that the mapping $\Theta$ is continuous if for every convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq E$ with limit point $x \in E$, the sequence of upper-semi-continuous functions $\{\Theta(x_n)\}_{n \in \mathbb{N}}$ converges towards the upper-semi-continuous function $\Theta(x)$, in the sense specified by Serra in [2, Theorem XII-2, p. 429]. In the remainder of this paper, the structuring function mapping $\Theta$ is assumed to be continuous from $E$ to $USC(E)$ and the support of $\Theta'(x)$ is assumed to be compact for every $x \in E$.

**Definition 1:** The spatially-variant function-processing (SVFP) erosion is given by

$$E_\Theta(f)(x) = \bigwedge_{u \in \text{Spt}(\Theta(x))} \{f(u) - [\Theta(u)](u)\} \quad (4)$$

$$\text{if Spt}(\Theta(x)) \subseteq \text{Spt}(f), \text{ and } -\infty, \text{ otherwise.} \quad (5)$$

**Definition 2:** The spatially-variant function-processing dilation is given by

$$D_\Theta(f)(x) = \bigvee_{u \in \text{Spt}(\Theta(x)) \cap \text{Spt}(\Theta'(x))} \{f(u) + [\Theta'(x)](u)\} \quad (6)$$

$$\quad \text{and} \quad \bigwedge_{v \in T : -\Theta'(x) + v \geq f}, \quad (7)$$

3If Spt(\Theta) is compact and Spt(\Theta(x)) is compact for all $x \in E$, then Spt(\Theta'(x)) is compact for all $x \in E$.

The SVFP erosion and dilation can be derived from their SVSP counterparts by characterizing the u.s.c. functions in terms of their umbrae [24].

The SVFP opening and closing are defined in the following obvious way:

**Definition 3:** The spatially-variant function-processing morphological opening is given by

$$\Gamma_\Theta(f) = D_\Theta(E_\Theta(f)) = \vee \{\Theta(u) + v \leq f ; (u, v) \in E \times T\} \quad (8)$$

The spatially-variant function-processing morphological closing is given by

$$\Phi_\Theta(f) = E_\Theta(D_\Theta(f)) = \wedge \{\Theta'(u) + v \geq f ; (u, v) \in E \times T\} \quad (9)$$

Observe that Eqs. (8) and (9) have a geometric interpretation that is analogous to their translation-invariant counterparts [8].

The properties of the SVFP erosion, dilation, opening and closing are investigated in Appendix A. In particular, we show that they satisfy the main properties of their translation-invariant counterparts [10], [21].

**C. Spatially-Variant Function-Set-Processing Systems**

Given a set $A \in \mathcal{P}(E)$, we denote by $C_A$ the characteristic function of $A$, i.e., $C_A(z) = 1$ if $z \in A$ and $C_A(z) = 0$ if $z \not\in A$. To each function-processing system $\Phi$, we associate its set-processing (SP) system $\phi$ defined as $\Phi(C_A) = C_{\Phi(A)}$. We say that $\Phi$ obeys the threshold superposition principle if

$$[\Phi(f)](x) = \vee \{t \in T : x \in \phi[X_t(f)]\} \quad (f \in USC(E)). \quad (10)$$

Thus, a function-processing system satisfying Eq. (10) transforms a function $f$ by decomposing it into its cross-sections and transforming each cross-section by the corresponding SP system. Such a system is called a function-set-processing system (FSP) by Maragos [5]. A sufficient condition for a FSP system to obey the threshold superposition property is to commute with thresholding [5], i.e.,

$$\phi[X_t(f)] = X_t[\Phi(f)], \quad (t \in T, f \in USC(E)). \quad (11)$$

The above condition allows us to analyze a function-processing system by looking at it as a set-processing system, which is simpler to analyze.

The following proposition shows that if $E_\Theta(X)$ (resp., $D_\Theta(X)$) is the SVSP erosion (resp., dilation) of the set $X \in \mathcal{P}(E)$ by the SV structuring element mapping $\theta : E \rightarrow \mathcal{P}(E)$ [11], and $f \in USC(E)$, then the sets $E_\Theta(X_t(f))$ (resp., $D_\Theta(X_t(f))$) satisfy the conditions to be the threshold sets of a function, $E_\phi(f)$ (resp., $D_\phi(f)$), defined as the SVFSP erosion (resp., dilation) of $f$ by the structuring element mapping $\theta$.

$$E_\phi(f)(x) = \wedge_{u \in \text{Spt}(f) \cap \Theta'(x)} f(u) \quad (f \in USC(E), x \in E), \quad (12)$$

and

$$D_\phi(f)(x) = \vee_{u \in \text{Spt}(f) \cap \Theta(x)} f(u) \quad (f \in USC(E), x \in E). \quad (13)$$
Proposition 1: We have

\[ E_\theta(X_t(f)) = X_t[\varepsilon_\theta(f)] \quad (f \in \text{USC}(E)). \] (14)

and

\[ D_\theta(X_t(f)) = X_t[D_\theta(f)] \quad (f \in \text{USC}(E)). \] (15)

Notice that if the mappings \( \theta \) and \( \theta' \) have finite range, (i.e., \( \theta(x) < \infty \) and \( \theta'(x) < \infty \), \( \forall x \in E \), where \( |X| \) denotes the cardinality of the set \( X \)), then the SVFSP erosion and dilation, defined in Eqs. (12) and (13), respectively, correspond to the adaptive minimum and maximum operators.

The SVFSP erosion and dilation of \( f \) by the SV structuring element mapping \( \theta \) are special cases of the SVFP erosion and dilation, defined in Eqs. (4) and (6), corresponding to the choices of the structuring element mapping \( \Theta(x) = g_x \) and \( \Theta'(x) = \bar{g}_x \), \( \forall x \in E \). In particular, \( \text{Spt}(\Theta(x)) \), \( \forall x \in E \). This case, we say that the structuring function mapping \( \Theta \) is flat. Observe that a flat structuring function mapping \( \Theta \) is represented by its region of support \( \Theta(x) = \text{Spt}(\Theta(x)), \forall x \in E \).

D. Examples

a) Translation-Invariant Gray-Level Morphology [21], [25]: Consider a function \( g \in \text{USC}(E) \). We showed in Section II-B that translation-invariant gray-level morphology corresponds to a structuring function mapping \( \Theta(x) = g_x \) and \( \Theta'(x) = \bar{g}_x \), \( \forall x \in E \). In particular, \( \text{Spt}(\Theta(x)) \) is compact if and only if \( \text{Spt}(g) \) is compact [9]. The SVFP erosion and dilation, defined in Eqs. (4)-(5) and (6)-(7), respectively, reduce to:

\[ f \ominus g(x) = \bigwedge_{u \in \text{Spt}(g)+x} \{f(u) - g(u-x)\} \] (16)

and

\[ f \oplus g(x) = \bigvee_{u \in \text{Spt}(g)+x} \{f(u) + g(u-x)\} \] (18)

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and

\[ f \oplus g(x) = \bigvee_{u \in \text{Spt}(g)+x} \{f(u) + g(u-x)\} \] (18)

Equations (16)-(17) and (18)-(19) are the well-known translation-invariant gray-level erosion and dilation, respectively [9], [10]. A similar derivation can be used to show that the SVFP opening and closing, defined in Eqs. (8) and (9), also reduce to their translation-invariant counterparts. Therefore, the translation-invariant gray-level morphology is a special case of the proposed spatially-variant gray-level morphology.

b) Gray-level Adaptive Neighborhood Morphology [18], [19]: Consider \( E = \mathbb{R}^2 \). Let \( h : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a criterion mapping such as luminance, contrast, thickness, etc. Let \( m > 0 \). For each \( x \in E \), define the connected set \( V^h_m(x) \) by \( V^h_m(x) = \{y : |h(y) - h(x)| \leq m\} \). Choose the flat structuring function mapping \( \Theta \) with the following region of support, \( \theta \):

\[ \theta(x) = \bigcup_{z \in E} \{V^h_m(z) : x \in V^h_m(z)\}. \] (20)

One can easily verify that the transposed region of support \( \theta' = \theta \) (i.e., \( \theta'(x) = \theta(x), \forall x \in E \)). Then, the SVFSP erosion and dilation, in Eqs. (12) and (13), become

\[ E_\theta(f) = \bigwedge_{u \in \theta(x)} f(u), \quad (f \in \text{USC}(E)), \] (21)

and

\[ D_\theta(f) = \bigvee_{u \in \theta(x)} f(u), \quad (f \in \text{USC}(E)). \] (22)

Equations (21) and (22) are, respectively, the adaptive neighborhood erosion and dilation presented in [18] and [19]. Therefore, adaptive neighborhood gray-level morphology framework is a special case of the proposed spatially-variant gray-level mathematical morphology theory.

c) Vertically-Invariant Morphology [17]: Given a structuring function mapping \( K : E \rightarrow \text{USC}(E) \), choose the structuring function mapping \( \Theta \), such that \( \Theta(x)[u] = [K(x)](u-x), \forall x, u \in E \). Observe that the local structuring function \( \Theta(x) \) at point \( x \), evaluated at \( x \), is equal to the local structuring function \( K(x) \) at point \( x \), evaluated at the origin, i.e., \( \Theta(x)(u) = [K(x)](0) \). For instance, consider the following structuring function mapping \( K [17]:

\[ [K(x)](u) = \begin{cases} \sqrt{r^2 - u^2}, & \text{if } |u| \leq r; \\ -\infty, & \text{if } |u| > r. \end{cases} \] (23)

Then,

\[ \Theta(x)(u) = \begin{cases} \sqrt{r^2 - (u-x)^2}, & \text{if } |u-x| \leq r; \\ -\infty, & \text{if } |u-x| > r. \end{cases} \] (24)

The local structuring function \( K(x) \) at the point \( x \) is a circle with radius \( r \) centered at the origin; whereas the local structuring function \( \Theta(x) \) at the point \( x \) is a circle with the same radius \( r \) but centered at the point \( x \). This property of having the local structuring function, \( \Theta(x) \), centered at \( x \), may be desired in some practical applications such as adaptive signal smoothing [17]. In this case, the SVFP erosion and dilation, defined in Eqs (4) and (6), reduce, respectively, to

\[ E_K [f](x) = \bigwedge_{z \in \text{Spt}(K(z))} \{f(x+z) - [K(x)](z)\}, \] (25)

and

\[ D_K [f](x) = \bigvee_{z \in \text{Spt}(f)+z \cap \text{Spt}(K'(z))+z} \{f(x-z) + [K(x-z)](z)\}, \] (26)

\[ \forall f \in \text{USC}(E), \forall x \in E. \]

Equations (25) and (26) coincide with the vertically-invariant erosion and dilation defined in [17] and used for adaptive signal smoothing. Therefore, vertically-invariant mathematical morphology provides another special case of the proposed spatially-variant gray-level morphology.

III. SPATIALLY-VARIANT KERNEL AND BASIS REPRESENTATIONS

A. SV Kernel Representation

Definition 4: A function-processing system \( \Psi : \text{USC}(E) \rightarrow \text{USC}(E) \) is called a V-system if \( \Psi(f+y) = \Psi(f) + y \), for all \( f \in \text{USC}(E) \) and \( y \in T \).

In particular, a V-system is invariant with respect to DC biases. Examples of V-systems are given by the SVFP erosion, dilation, opening and closing, defined in Section II-B, adaptive neighborhood morphological systems defined in Eqs. (21) and (22), adaptive amoeba morphological systems presented in [20] and the vertically invariant morphological systems defined in Eqs. (25) and (26). Moreover, one can easily verify that the
class of V-systems is closed under duality, i.e., if $\Psi$ is a V-system, then its dual $\Psi^*$ is also a V-system.

We extend Maragos’ kernel representation theorem to V-systems as follows: Let $\Psi$ be a V-system and consider its SV umbra processing system $\psi_n$ defined as $\psi_n(U[f]) = U[\Psi(f)]$, for every $f \in USC(E)$. From the definition of the kernel of SVSP systems in [11], the proof of the following proposition derives a one-to-one correspondence between the kernel of a V-system and the kernel of its umbra processing system.

**Proposition 2:** The kernel of a V-system $\Psi$, $K(\Psi)$, is given by the following collection of mappings:

$$K(\Psi) = \{ \Theta : E \rightarrow USC(E) : \Psi[\Theta(x)](x) \geq 0, \forall x \in E \}.$$  \hspace{1cm} (27)

In this paper, we use $K$ to denote the kernel of a SVFP system and $Ker$ to denote the kernel of a SVSP system [11]. The one-to-one correspondence between the kernel of a V-system and the kernel of its SV umbra set-processing system will allow us to transpose the results of the kernel representation obtained for SVSP systems in [11] to V-systems. In particular, since the kernel of SVSP systems is non-trivial and unique [11, Propositions 1 and 2], it follows that the kernel of V-systems is also non-trivial and unique.

Using Eqs. (2), (10) and the definition of the kernel of SVSP systems in [11], we obtain the kernel of a SVFP system $\Phi$ from the kernel of its SP system $\phi$ as follows:

$$K(\Phi) = \{ \Theta : \text{as infimum of spatially-variant function-processing dilations} \}.$$  \hspace{1cm} (28)

The following theorem establishes the kernel representation for increasing V-systems.

**Theorem 1:** A spatially-variant function-processing system $\Psi : USC(E) \rightarrow USC(E)$ is an increasing V-system if $\Psi$ can be represented as supremum of spatially-variant function-processing erosions by mappings in its kernel or equivalently as infimum of spatially-variant function-processing dilations by the transposed mappings in the kernel of its dual, i.e.,

$$\Psi(f) = \bigvee_{\Theta \in K(\Psi)} \Theta \in USC(E).$$  \hspace{1cm} (29)

**Corollary 1:** A spatially-variant function-set-processing system $\Phi : USC(E) \rightarrow USC(E)$ that commutes with thresholding is an increasing V-system if it can be represented as supremum of spatially-variant function-set-processing erosions by mappings in the kernel of its set-processing system $\phi$ or equivalently as infimum of spatially-variant function-set-processing dilations by the transposed mappings in the kernel of $\phi^*$;

$$\Phi(f) = \bigvee_{\Theta \in Ker(\phi)} \Theta \in USC(E).$$  \hspace{1cm} (30)

**B. Basis Representation**

The kernel representations in theorem 1 and corollary 1 are powerful theoretical results as they demonstrate the ubiquity of the SVFP and the SVSP erosion and dilation and hence establish spatially-variant mathematical morphology for gray-level signals as the general mathematical framework for the study of linear and non-linear increasing systems in signal and image processing. However, the kernel representation theorems have no direct relevance for practical implementation of V-systems because of the infinite cardinality of the kernel. To see this, consider an increasing V-system $\Psi$. From Eq. (27), we observe that if $\Theta \in K(\Psi)$, then every structuring function mapping $\Lambda \geq \Theta$ is also in the kernel of $\Psi$. Therefore, we are lead to the investigation of the existence of the minimal kernel elements for the representation of V-systems. Following Maragos’ approach in defining the basis of translation-invariant set-processing and function-processing systems [5], we define the basis, $\mathcal{B}_\Psi$, of a V-system $\Psi$ as the collection of the minimal elements of the kernel of $\Psi$. Formally,

$$\mathcal{B}_\Psi = \{ \Theta \in K(\Psi) : [\Lambda \in K(\Psi) \land \Lambda \leq \Theta] \implies \Lambda = \Theta \}.$$  \hspace{1cm} (31)

In this paper, we use $\mathcal{B}_\Psi$ to denote the basis of the SVFP $\Psi$ and $\mathcal{B}_{\Psi^*}$ to denote the basis of the SVSP system $\Psi^*$ defined in [11]. In the development of the basis representation for SVSP systems in [11], we had to restrict ourselves to the class of all closed subsets of $E$. The equivalent class of functions is the class of upper-semi-continuous functions. In fact, a function $f$ is upper-semi-continuous iff its umbra $U[f]$ is closed or equivalently iff its threshold sets $\chi_t(f)$ are closed for all $t \in T$. Let $f_n \downarrow f$ be a sequence of u.s.c. functions that decrease monotonically to $f = \wedge_n f_n$. An increasing function-processing system $\Psi$ is said to be upper-semi-continuous if and only if $f_n \downarrow f \implies (\Psi(f_n) \downarrow \Psi(f))$.

In the following theorem, we prove that every increasing upper-semi-continuous V-system has a minimal element in its kernel.

**Theorem 2:** Let $\Psi : USC(E) \rightarrow USC(E)$ be an increasing upper-semi-continuous V-system. Then the kernel of $\Psi$ has a minimal element.

Before we prove the potential of minimal elements for exact representation of these systems, we need the following result:

**Theorem 3:** Let $\Psi$ be an increasing upper-semi-continuous V-system. Then for every $\Theta \in K(\Psi)$, there exists $\Theta_M \in \mathcal{B}_\Psi$ such that $\Theta_M \leq \Theta$.

We can now establish the basis representation of upper-semi-continuous V-systems in terms of SVFP erosions.

**Theorem 4:** Let $\Psi : USC(E) \rightarrow USC(E)$ be an increasing upper-semi-continuous V-system. Then $\Psi$ can be represented as supremum of spatially-variant function-processing erosions by mappings in its basis $\mathcal{B}_{\Psi^*}$;

$$\Psi(f) = \bigvee_{\Theta \in \mathcal{B}_{\Psi^*}} \Theta \in USC(E).$$  \hspace{1cm} (32)

To find a dual representation in terms of SVFP dilations, Theorem 4 has to apply to the dual V-system $\Psi^*$. In particular, $\Psi^*$ has to be upper semi-continuous on $USC(E)$. Consequently, the class $USC(E)$ has to be invariant under function complementation, i.e., if $f$ is u.s.c. then $(-f)$ is also u.s.c. This is in particular true for functions defined on $\mathbb{Z}^m$. In this case, both $\Psi$ and $\Psi^*$ are defined on $USC(\mathbb{Z}^m)$.

**Corollary 2:** Let $\Psi : USC(\mathbb{Z}^m) \rightarrow USC(\mathbb{Z}^m)$ be an increasing and upper-semi-continuous V-system. If the dual system $\Psi^*$ is also upper-semi-continuous, then $\Psi$ can be represented as supremum of erosions by mappings in its basis or equivalently as infimum of dilations by the transposed
mappings in the basis of its dual, i.e.,

\[ \Psi(f) = \bigvee_{\theta \in B_0} \mathcal{E}_\theta(f) = \bigwedge_{\theta \in B_0^s} \mathcal{D}_\theta(f) \quad (f \in USC(\mathbb{Z}^m)). \tag{33} \]

**Corollary 3:** a) Let \( \Phi : USC(\mathbb{E}) \rightarrow USC(\mathbb{E}) \) be a spatially-variant function-set-processing system that commutes with thresholding, then \( \Phi \) can be represented as supremum of erosions by mappings in the basis of its set-processing system \( \Phi^* \);

\[ \Phi(f) = \bigvee_{\theta \in B_0} \mathcal{E}_\theta(f) \quad (f \in USC(\mathbb{E})). \tag{34} \]

b) Let \( \Phi : USC(\mathbb{Z}^m) \rightarrow USC(\mathbb{Z}^m) \) be a spatially-variant function-set-processing system that commutes with thresholding, and consider its set-processing system \( \phi \). If the dual set-processing system \( \phi^* \) is upper-semi-continuous, then \( \Phi \) can be represented as supremum of erosions by mappings in the basis of its set-processing system \( \phi \) or equivalently as infimum of dilations by the reflected mappings in the basis of \( \phi^* \), i.e.,

\[ \Phi(f) = \bigvee_{\theta \in B_0} \mathcal{E}_\theta(f) = \bigwedge_{\theta \in B_0^s} \mathcal{D}_\theta(f) \quad (f \in USC(\mathbb{Z}^m)). \tag{35} \]

The perspectives of the basis theory are at least two-fold. First, the redundancy of the kernel is infinitely reduced. Second, if the basis is finite, the corresponding V-system can be represented as maximum of SVFP erosions or as minimum of SVFP dilations. This can tremendously simplify the analysis and the implementation of these systems. However, the basis representation theorem is not constructive, in the sense that it does not provide an algorithm to find the basis elements for each increasing and upper-semi-continuous V-system. It is merely an existence theorem. In the following section, we present examples of practical V-systems used in signal and image processing applications, and show how to obtain their basis.

**IV. EXAMPLES**

A. Order-Statistic Filters

In this example, we study the properties of the SVFSP order-statistic filters and show their relation to the SVFSP basic morphological systems.

Consider \( E \subseteq \mathbb{Z}^2 \). Let \( B \) be a mapping from \( E \) into \( \mathcal{P}(E) \) such that \( y \in B(y) \) and \( |B(y)| = \text{cardinality of } B(y) = n \), for every \( y \in E \). The \( r \)-th SVFSP order-statistic filter is defined by:

\[ [\Phi_{\mathcal{E}}(f,B)](x) = r^\text{th} \text{ largest value of } \{ f(y) : y \in B(x) \}. \tag{36} \]

where \( r = 1, 2, \ldots, n \). Observe that the SVFSP 1\textsuperscript{st} order-statistic is the SVFSP dilation by the transposed mapping \( B_0 = \{ -b : b \in B \} \); and the SVFSP \( n \)-th order-statistic is the SVFSP erosion by the mapping \( B \).

**Proposition 3:** The \( r \)-th SVFSP order-statistic filter is an increasing V-system, which commutes with thresholding. Moreover, its dual is the \( (n-r+1) \)-th SVFSP order-statistic filter.

So, from corollary 3-b, the \( r \)-th SVFSP order-statistic filter can be represented as supremum of SVFSP erosions by mappings in the kernel of its SP system or equivalently as infimum of SVFSP dilations by the mappings in the kernel of the dual SP system. However, this representation is not very useful in practice because of the redundancy of the kernel.

**B. Linear Time-Varying Systems**

In this example, we generalize the study of linear time invariant systems in [9] to linear time-varying (LTV) systems. This example will show the power of the proposed spatially-variant gray-level mathematical morphology theory to study not only SV non-linear systems but also SV linear systems as well.

The output of a continuous linear time-varying (LTV) system is given by

\[ [\Psi(f)](t) = \int_{\mathbb{R}} f(\tau)[h(\tau)](t) d\tau, \tag{39} \]

where \( h(\tau) \) is the response of the system to an impulse applied at time \( \tau \). Therefore, the impulse response mapping of a linear time-varying system can be viewed as a mapping from \( \mathbb{R} \) to \( Func(\mathbb{R}) \) such that for each impulse applied at time \( \tau \in \mathbb{R} \) corresponds an impulse response \( h(\tau) \in Func(\mathbb{R}) \). Without loss of generality, we consider systems such that their degain is equal to unity, i.e., \( \int_{\mathbb{R}} [h(\tau)](t) d\tau = 1 \), for all \( t \in \mathbb{R} \). Observe that this scaling condition ensures that the linear time-varying system \( \Psi \), defined in Eq. (39), is a V-system. In the following proposition, we give a necessary and sufficient condition for \( \Psi \) to be increasing.

**Proposition 4:** A linear time-varying system is increasing iff its impulse response mapping is non-negative, i.e., for each \( \tau \), \( h(\tau) \) is a non-negative function.

The kernel of the linear time-varying system \( \Psi \), defined in Eq. (39), is given by

\[ \mathcal{K}(\Psi) = \{ \Theta : \int_{\mathbb{R}} [\Theta(x)](\tau)[h'(x)](\tau) d\tau \geq 0, \forall x \in \mathbb{R} \}, \tag{40} \]

where \( h' \) is the transpose mapping of \( h \) (see Eq. (3)). So, from Theorem 1, we obtain a kernel representation of the linear time-varying system as follows:

**Proposition 5:** Let \( \Psi \) be a linear time-varying system with unity gain and a non-negative impulse response mapping. Then

\[ [\Psi(f)](x) = \int_{\mathbb{R}} f(\tau)[h(\tau)](x) d\tau \]

\[ = \bigvee_{\Theta \in \mathcal{K}(\Psi)} \bigwedge \{ f(u) - [\Theta(x)](u) \}. \tag{41} \]

Proposition 5 gives a closed form expression of the output of a linear time-varying system having unity gain and a
non-negative impulse response mapping in terms of sup-inf operations only. The drawback of this expression is that there are an infinite number of such operations since the kernel of the system is infinite.

In what follows, we will investigate the existence of a basis representation for discrete linear time-varying systems. A sufficient condition for a discrete LTV system with a non-negative impulse response mapping to be upper-semi-continuous is that the transposed mapping $h'(n)$ has a finite support for all $n$, where the support of the function $h(k)$, $\text{Spt}(h(k))$, is defined as $\text{Spt}(h(k)) = \{ n \in \mathbb{N} : [h(k)](n) \neq 0 \}$. One can easily verify that, in this case, $f_n \downarrow f = \Psi(f_n) \uparrow \Psi(f)$.

**Proposition 6:** Let $\Psi$ be a discrete linear time-varying system with unity gain and a non-negative impulse response mapping having a transposed mapping with finite support. Then the basis of $\Psi$ is given by

$$
\mathfrak{B}_\Psi = \{ \Theta : \sum_{k=1}^{N} [h'(n)][k][\Theta(k)][(n)] = 0, \quad \text{and} \quad \text{Spt}(\Theta(n)) = \text{Spt}(h'(n)), \forall n \in \mathbb{N} \}.
$$

If we represent the finite extent function $h'(n)$ in a vector form, then we see that the basis elements $\Theta(n)$ belong to the hyperplane perpendicular to the vector $h'(n)$, for all $n \in \mathbb{N}$. The basis mappings are solutions of the linear system $\sum_{k=1}^{N} [h(k)](n)[\Theta(k)] = 0, \forall n \in \mathbb{N}$ subject to three constraints: 1) $h(k)$ is a non-negative function for $k = 1, \ldots, N$; 2) $\sum_{k=1}^{N} [h(k)](1) = 1, \forall n \in \text{Spt}(h(k));$ 3) $\text{Spt}(\Theta(n)) = \text{Spt}(h'(n))$ is finite, $\forall n$. Consequently, we have the following basis representation

$$
[\Psi(n)](n) = \sum_{k=1}^{N} f(k)[h(k)](n)
$$

Eq. (42) relates the non-linear SVFP morphological erosions to LTV systems. Moreover, it gives a closed-form expression to a large class of linear time-varying discrete systems.

**Example:** Adaptive mean: Consider a normalized 1-D signal $x(n)$ and its corrupted version by an impulse noise, $z(n)$, i.e.,

$$
z(n) = x(n) + \sum_{k \in I} (-1)^k \delta(n-k),
$$

where $I \subset \mathbb{N}$ is the set of corrupted samples. We propose to adaptively denoise the signal $z(n)$ by using a LTV system with the following causal impulse response mapping:

$$
[h(k)](n) = \begin{cases} 
\delta(n-k), & \text{if } n \notin I; \\
\frac{1}{2}[\delta(n-k) + \delta(n-1-k)], & \text{if } n \in I;
\end{cases}
$$

where $k \in \{n-1,n\}$ and $h(k) = 0$ if $k \notin \{n-1,n\}$. The output of this filter is given by

$$
y(n) = \sum_{k=n-1}^{n} z(k)[h(k)](n)
$$

From Eq. (45), we see that this LTV filter is an adaptive mean, that averages the last two samples of the signal if the current sample is noisy and leaves the current sample unchanged if it is not noisy. This simple strategy, illustrated in Fig. 1, denoises the signal without oversmoothing it.

The impulse response mapping, defined in Eq. (44), is non-negative, has a unity dc gain and the support of its transpose mapping is finite. Therefore, from Proposition 6, the basis functions are given by

$$
\Theta(n) = \begin{cases} 
0, & \text{if } n \notin I; \\
\left[ \frac{(\Theta(n))(n)}{(\Theta(n))(n-1)} \right] = \left[ \frac{\alpha}{\alpha}, \right] & \text{if } n \in I,
\end{cases}
$$

where $\alpha \in \mathbb{R}$. Thus, from Eq. (42), the adaptive mean filter can be represented as a supreme of minima:

$$
y(n) = \begin{cases} 
x(n), & \forall \alpha \in \mathbb{R} \min\{x(n)-\alpha, x(n-1)+\alpha\}, & \text{if } n \notin I; \\
x(n), & \text{if } n \in I.
\end{cases}
$$

V. SIMULATIONS

A. Adaptive Denoising

Image restoration is an important problem in image processing and analysis applications. It requires the development of an efficient filtering procedure that restores an image from its noisy version while preserving the important features of the noise-free image. This is an important requirement since many algorithms for pattern analysis, which process noisy data, critically depend on accurate geometrical and topological image description [2]. The traditional approach to solving this problem is by means of linear filtering techniques. Although this is a mathematically and practically simple approach, it usually results in a distortion of many important image characteristics. The alternative solution is by means of more powerful non-linear filtering techniques, and specifically, by employing the class of morphological filters [9], [26]. However, it is known that there is an inherent tradeoff in translation-invariant morphological filters between the noise removal capability of the filter and the feature preservation of the noise-free image [16], [17], [27]. This tradeoff is due to the use of a fixed SE while morphologically filtering the signal (or image). The important structures of the signal that are smaller than the SE used will be removed or oversmoothed. One solution to the denoising problem then, is to vary the SE according to the local characteristics of the image. In this example, we will show the power of linear and non-linear spatially-variant denoising by considering the SV mean filter, the SV alternating filter [28], [2], and the SV median filter.

Consider the corrupted Lena image by a 10% salt (gray-level 0) and pepper (gray-level 255) noise, shown in Fig. 1(a). Let $B$ be a fixed SE and consider a flat structuring function mapping with a region of support mapping, $\theta$, given by

$$
\theta(x) = \begin{cases} 
B, & \text{if } x \text{ is a salt or pepper pixel;} \\
\emptyset, & \text{otherwise.}
\end{cases}
$$

That is only noisy pixels are filtered. A noisy pixel is detected as an isolated 0 or 255 gray-level pixel. This SV denoising scheme will significantly preserve the edges while effectively removing the noise. In our simulations, $B$ is a square window of a pre-determined size.
d) SV Mean filter: The usual mean filter is a linear, simple and easy to implement system, often used for image smoothing and denoising. There is an inherent tradeoff in the choice of the window of the mean filter: A small window preserves to an extent the edges of the image but is sensitive to outliers whereas a large window reduces the effect of outliers but significantly blurs the edges of the image. This tradeoff is illustrated in Figs. 1(b) and 1(d). We adaptively mean filter the noisy image using a 2-D version of the impulse response mapping given in Eq. (44). The power of the adaptive mean filter in denoising is illustrated in Figs. 1(c) and 1(e).

e) SV alternating filter: The alternating filter is a composition of closing and opening by the same SE. Maragos and Schafer [26] have demonstrated a strong relationship between the alternating filter and the median filter. The alternating filter has been experimentally demonstrated for its smoothing and noise removal capability in binary and gray-scale images [2], [28], [29]. Figure 1(f) shows the translation-invariant alternating filter output, with a $3 \times 3$ square SE. Some salt noise remains and the image is overly smoothed. The SV alternating filter, displayed in Fig. 1(g), removes all the noise and preserves the edges of the noise-free image.

f) SV median filter: The median filter is a self-dual rank order filter (see Section IV). The translation-invariant median filter is more robust than the translation-invariant mean filter in removing the noise. But it is also relatively expensive and complex to compute as it requires sorting algorithms. Moreover, the translation-invariant median filter removes the noise at the expense of over-smoothing the image as shown in Fig. 1(h). On the other hand, the SV median filter preserves the noise-free image features as can be seen from Fig. 1(i).

B. SV multiscale filtering and segmentation

In this application, we will show the power of SV gray-level mathematical morphology in multiscale filtering and segmentation by presenting a multiscale representation of the cameraman image using the Alternating Sequential Filters (ASF) and segmenting the filtered images using the watershed transformation. An alternating sequential filter is a composition of openings and closings by structuring elements of increasing sizes. The alternation of openings and closings is essentially a multisresolution technique, which introduces less distortion than individual openings and closings. Schonfeld and Goutsias showed that alternating sequential filters are the best filters in preserving crucial structures in the “least difference” sense [30].

The watershed transformation is a powerful tool for image segmentation [8], [31]. The intuitive idea underlying this method is that of a landscape or topographic relief which is flooded by water, watersheds being the divide lines of the domains of attraction of rain falling over the region. An alternative approach is to imagine the landscape being immersed in a lake, with holes pierced in local minima. The water entering through the holes floods the surface. When two or more floods coming from different minima may merge, dams are built. At the end of the process, only the dams emerge. These dams define the watershed lines. In order to produce a meaningful segmentation, the input image is generally transformed, then the watershed is applied. The gradient image is often used in the watershed transformation, because the main criterion of the segmentation is the homogeneity of the gray values of the objects present in the image. However, the gradient image generally creates an over-segmentation which is due to the presence of spurious minima. In this simulation, we show that applying the watershed transformation to a spatially-variant ASF (SVASF) produces better segmentations than applying it to the gradient image or to the translation-invariant counterpart filter (TIASF).

In our implementation, we use balls SE’s of increasing radius for the TIASF, and the flat structuring function mapping represented by its region of support given in Eq. (20), for the SVASF. We write TIASF$_p$, to denote the TIASF of order $p$ and SVASF$_m$ to denote the SVASF of order $p$, homogeneity $m$ and criterion mapping given by the luminance. Figure 2 shows the decomposition and segmentation results of the different filters. The translation-invariant ASF rapidly oversmoothes the image altering the transitions between the different objects and loosing the original topology of the image. Even though it results in a less mosaic segmented image than the gradient, it loosens the oversmoothed objects (see Fig. 2(g)-(h) where the camera stand is not represented) and still results in an over-segmentation of the background. The SVASF, however, results in a simplified version of the image while conserving the topology and contours of its different objects and at the same time producing flat zones of the image, which lead to a much better segmentation than its translation-invariant counterpart.

VI. Summary

We proposed a spatially-variant gray-level mathematical morphology theory in the Euclidean space, which preserves the geometrical notion of the structuring function inherent in the classical translation-invariant morphology. We defined the basic spatially-variant gray-level morphological operations, i.e., spatially-variant erosion, dilation, opening, and closing, and investigated their properties. We have demonstrated the ubiquity of spatially-variant function-processing (SVFP) erosions and dilations by showing that every increasing V-system, i.e., an increasing system that is invariant under vertical translations, can be represented as supremum of SVFP erosions or, equivalently, as infimum of SVFP dilations. Furthermore, we established a basis representation for the subclass of upper-semi-continuous increasing V-systems. If the basis of a system is finite, then it can be represented as maximum of SVFP erosions or minimum of SVFP dilations. In particular, we showed that adaptive order-statistic filters have a basis representation in terms of SVFP erosions and SVFP dilations. We have also related linear time-varying (LTV) systems to the basic non-linear SVFP morphological operators. In particular, we established a closed form expression for LTV systems in terms of supremum and infimum of functions. Simulation results showed the enormous potential of the theory of spatially-variant gray-level mathematical morphology in image denoising and multiscale representation.
APPENDIX A: PROPERTIES OF THE BASIC SVFP MORPHOLOGICAL OPERATORS

A. Properties of SVFP erosion and dilation

The below properties are valid for all functions $f \in USC(E)$.

g) Adjunction: For every structuring function mapping $\Theta$, the pair $(E_\Theta, D_\Theta)$ is an adjunction, i.e.,

$$D_\Theta(f) \leq g \iff f \leq E_\Theta(g).$$

Proof: We have

$$D_\Theta(f) \leq g \iff \forall x \in E, \forall u \in E \{f(u) + [\Theta(u)](x)\} \leq g(x)$$

$$\iff \forall x, u \in E, f(u) + [\Theta(u)](x) \leq g(x)$$

$$\iff \forall u \in E, f(u) \leq \wedge_{x \in E} \{g(x) - [\Theta(u)](x)\}$$

$$\iff \forall u \in E, f(u) \leq E_\Theta(g)(u)$$

$$\iff f \leq E_\Theta.$$

h) Duality: For every structuring function mapping $\Theta$, the spatially-variant function-processing systems $E_\Theta$ and $D_\Theta$ are dual, i.e., $E_\Theta^* = D_\Theta$.

Proof: We have

$$E_\Theta^*(f) = -E_\Theta(-f) = -\vee \{v \in T : \Theta(x) + v \leq -f\}$$

$$= \wedge \{-v \in T : f \leq -v - \Theta(x)\}$$

$$= \wedge \{v \in T : f \leq v - \Theta(x)\} = D_\Theta^*(f).$$

i) Increasing: For every structuring function mapping $\Theta$, the spatially-variant function-processing systems $E_\Theta$ and $D_\Theta$ are increasing systems.

Proof: The proof follows immediately from Eqs. (4) and (6).
Fig. 2. Translation-invariant and spatially-variant multiscale decomposition using Alternating Sequential Filters (ASF), and segmentation using the watershed transformation: (a) The original cameraman image; (b) Segmentation of the original image; (c) Segmentation of the gradient image; (d)-(f) Translation-invariant ASF; (g)-(i) Segmentation of the results in (d)-(f), respectively; (j)-(l) spatially-variant ASF with a homogeneity tolerance $m = 10$; (m)-(o) Segmentation of the results in (j)-(l), respectively.
B. Properties of the SVFP opening and closing

From the properties of the SVFP erosion and dilation, it follows that the SVFP opening and closing are increasing dual operators. Moreover, the SVFP opening is anti-extensive and the SVFP closing is extensive.

j) Extensivity and anti-extensivity: If \( \Theta(x) \geq 0 \), \( \forall x \in E \), then

\[
E_{\Theta}(f) \leq f \quad \text{and} \quad D_{\Theta}(f) \geq f. \tag{49}
\]

Proof:

\[
E_{\Theta}(f)(x) = \bigvee_{u \in \text{Spt}(\Theta(x))} \{ f(u) - |\Theta(x)|[u] \} \leq f(x) - |\Theta(x)|[x] \leq f(x),
\]

where the last inequality follows from the fact that \( |\Theta(x)|[x] \geq 0 \). A similar argument can be used to show the extensivity of the SVFP dilation.

k) Scaling with respect to the spatially-variant structuring function mapping:

Proposition 7: If \( \Theta_1 \leq \Theta_2 \), then

\[
E_{\Theta_2}(f) \leq E_{\Theta_1}(f) \quad \text{and} \quad D_{\Theta_1}(f) \leq D_{\Theta_2}(f).
\]

Proof: Since, \( \Theta_1 \leq \Theta_2 \), we have for all \( x \in E \) and \( v \in T, \Theta_1(x) + v \leq \Theta_2(x) + v \). So, for a given \( f \in \text{USC}(E) \), we have \( \{ v : \Theta_2(x) + v \leq f \} \subseteq \{ v : \Theta_1(x) + v \leq f \} \). Hence, \( \bigvee \{ v : \Theta_2(x) + v \leq f \} \leq \bigvee \{ v : \Theta_1(x) + v \leq f \} \) or equivalently \( E_{\Theta_2}(f) \leq E_{\Theta_1}(f) \).

The increasing property of the SVFP dilation with respect to the structuring function mapping can be derived using similar arguments and the fact that \( \Theta_1 \leq \Theta_2 \Leftrightarrow \Theta'_1 \leq \Theta'_2 \).

l) Serial composition: Consider two structuring function mappings \( \Theta_1 \) and \( \Theta_2 \). We use \( E_{\Theta_1}(\Theta_2) \) and \( D_{\Theta_1}(\Theta_2) \) to denote the structuring function mapping given by \( E_{\Theta_1}(\Theta_2)(x) = E_{\Theta_1}(\Theta_2(x)) \) and \( D_{\Theta_1}(\Theta_2)(x) = D_{\Theta_1}(\Theta_2(x)) \), \( \forall x \in E \). We have

\[
E_{\Theta_2}[E_{\Theta_1}(f)] = E_{D_{\Theta_1}(\Theta_2)}(f), \tag{50}
\]

and

\[
D_{\Theta_2}[D_{\Theta_1}(f)] = D_{D_{\Theta_1}(\Theta_2)}(f). \tag{51}
\]

Proof: We have

\[
E_{\Theta_2}[E_{\Theta_1}(f)](x) = \bigvee \{ v : [\Theta_2(x)](u) + v \leq [E_{\Theta_1}(f)](u), \forall u \} = \bigvee \{ v : [\Theta_2(x)](u) + v \leq \bigvee \{ f(t) - |\Theta_1(u)|[t], \forall u \} \}
\]

Hence, we have \( \Gamma_\Theta(D_{\Theta(g)}) = D_{\Theta(g)} \), which is equivalent to \( \Gamma_\Theta(f) = f \). Therefore, we obtain a characterization of \( \Theta \)-open functions. A similar argument can be used to obtain a characterization of \( \Theta \)-closed functions.

APPENDIX C: PROOF OF PROPOSITIONS

Proof: [Proof of proposition 1] Consider \( f \in \text{USC}(E) \) and \( t \in T \). We have

\[
X_t[E_{\Theta}(f)] = \{ z : \bigwedge_{u \in \text{Spt}(\Theta(z))} \{ f(u) \geq t \} \}
\]

m) Idempotence: For every structuring function mapping \( \Theta \), the spatially-variant function-processing morphological opening and closing are idempotent, i.e.,

\[
\Gamma^2_{\Theta} = \Gamma_{\Theta}, \quad \text{and} \quad \Phi^2_{\Theta} = \Phi_{\Theta}. \tag{52}
\]

Proof:

\[
\Gamma_\Theta[\Gamma_\Theta(f)] = \bigvee \{ \Theta(u) + v \leq \bigvee \{ \Theta(a) + b \leq f \} \}
\]

A similar argument can be used to obtain the idempotence of the SVFP closing.

It was shown in [1] and [32] that the translation-invariant set-processing opening and closing can be exactly specified from their fixed points. In the following proposition, we provide a characterization of \( \Theta \)-open and \( \Theta \)-closed functions, which are the fixed points of the SVFP opening and closing, respectively.

Definition 5: A function \( f \) is \( \Theta \)-open (resp. \( \Theta \)-closed) if \( \Gamma_\Theta(f) = f \) (resp. \( \Phi_\Theta(f) = f \)).

A useful characterization of \( \Theta \)-open and \( \Theta \)-closed functions is given by the following proposition;

Proposition 8: A function \( f \) is \( \Theta \)-open (resp. \( \Theta \)-closed) if and only if there exists a function \( g \) such that \( f = D_{\Theta}(g) \) (resp. \( f = E_{\Theta}(g) \)).

Proof: Assume first that \( \Gamma_\Theta(f) = f \). Take \( g = E_{\Theta}(f) \). Then we have \( f = D_{\Theta}(g) \). Assume now that \( f = D_{\Theta}(g) \) for some function \( g \). By the anti-extensivity of the SVFP opening and the increasing property of the SVFP dilation, we have

\[
D_{\Theta}(g) \geq \Gamma_\Theta(D_{\Theta}(g)) = D_{\Theta}(E_{\Theta}(D_{\Theta}(g))) = D_{\Theta}(\Phi_\Theta(g)) \geq D_{\Theta}(g).
\]

Hence, we have \( \Gamma_\Theta(D_{\Theta}(g)) = D_{\Theta}(g) \), which is equivalent to \( \Gamma_\Theta(f) = f \). Therefore, we obtain a characterization of \( \Theta \)-open functions. A similar argument can be used to obtain a characterization of \( \Theta \)-closed functions.
If $\text{Spt}(\Theta'(x))$ is compact for all $x \in \mathbb{E}$, then the supremum on the set $\text{Spt}(\Theta'(x))$ is achieved. Then, we have

$$\mathcal{X}_U[\mathcal{D}_\Theta(f)] = \{ z : \sup_{u \in \text{Spt}(\Theta'(x))} \{ f(u) \geq t \} \}$$

$$= \{ z : \exists u \in \text{Spt}(\Theta) \cap \text{Spt}(\Theta'(x)), f(u) \geq t \}$$

$$= \{ z : \exists u \in \text{Spt}(\Theta) \cap \Theta'(x), f(u) \geq t \}$$

$$= \{ z : \Theta'(x) \cap \mathcal{X}(f) \neq \emptyset \}$$

$$= \mathcal{D}_\Theta[\mathcal{X}(f)].$$

**Proof:** [Proof of proposition 2] Let $\Psi$ be a $V$-system and consider its umbra processing system $\psi_u$. Define the umbra SE mapping $\Theta^U$ from $\mathbb{E}$ to the set of all umbra $\mathbb{E} \times T$, by $\Theta^U(x, y) = U[\Theta(x) + y] = U[\Theta(x)] + y$. $\psi_u$ has a SV kernel, $\text{Ker}(\psi_u)$, defined in [11]. We have

$$\Theta^U \in \text{Ker}(\psi_u)$$

$$\iff (x, y) \in \psi_u(\Theta^U(x, y)), \forall (x, y) \in \mathbb{E} \times T$$

$$\iff (x, y) \in \psi_u(U[\Theta(x) + y]), \forall (x, y) \in \mathbb{E} \times T$$

$$\iff (x, y) \in U[\Phi'(x, y)], \forall (x, y) \in \mathbb{E} \times T$$

$$\iff [\Psi(x)](x) \geq y, \forall (x, y) \in \mathbb{E} \times T$$

$$\iff [\Psi(x)](x) \geq y, \forall (x, y) \in \mathbb{E} \times T$$

$$\iff \Psi'(x)(x) \geq y, \forall x \in \mathbb{E}$$

$$\iff \Theta \in \mathcal{K}(\Psi).$$

**Proof:** [Proof of proposition 3]

a) Increasing: Let $f \leq g$. In particular, $f(y) \leq g(y) \forall y \in B(x)$ and $\forall x \in \mathbb{E}$. This implies that $\forall x \in \mathbb{E}, [\Phi_r(f, B)](x) \leq [\Phi_r(g, B)](x)$. Thus $\Phi_r$ is an increasing system.

b) Duality: Let $f \in \mathcal{U}SC(\mathbb{E})$. For every $x \in \mathbb{E}$, we have

$$[\Phi^*_r(f, B)](x) = [-\Phi(-f, B)](x)$$

$$= -\text{rth largest value of} \{ -f(y), y \in B(x) \}$$

$$= -\text{rth largest value of} \{ f(y), y \in B(x) \}$$

$$= \{ \Phi_{n-r+1}(f, B) \}(x).$$

Hence, $\Phi^*_r(\cdot, B) = \Phi_{n-r+1}(\cdot, B)$.

c) Commuting with thresholding: Consider a function $f \in \mathcal{U}SC(\mathbb{E})$. We have

$$z \in \mathcal{X}(\Phi_r(f, B))$$

$$\iff \{ \Phi_r(f, B) \}(z) \geq t$$

$$\iff \text{rth largest value of} \{ f(y), y \in B(z) \} \geq t$$

$$\iff \mathcal{X}_r(f, B) \supseteq z \iff z \in \Phi_r(\mathcal{X}_r(f, B)).$$

So $\mathcal{X}(\Phi_r(f, B)) = \Phi_r(\mathcal{X}_r(f, B))$. From Eq. (11), we conclude that $\Phi_r$ commutes with thresholding.

**Proof:** [Proof of proposition 4] Assume first that the function $h(\tau) \geq 0$, for all $\tau \in \mathbb{E}$. Consider two functions $f$ and $g$ such that $f \geq g$. Then $\Psi(f) - \Psi(g)(f) = (f - g)(\tau)h(\tau)(\tau)d\tau \geq 0$ since the integrand is a non-negative function. Hence, the LTV system $\Psi$ is increasing. Assume now that $\Psi$ is increasing, i.e., $f \leq g \Rightarrow \Psi(f) \leq \Psi(g).$ Thus $\int R(\tau)[h(\tau)](\tau)d\tau \geq 0$ for every non-negative function $R$. Let $p_k(x)$ be a sequence of triangular functions such that their width goes to zero and their height to $+\infty$ satisfying $\int p_k(x)dx = 1$, for all $k \in \mathbb{N}$. For a given $\tau$, we have $\forall t,$

$$[h(\tau)](t) = \int [h(\tau)](z)\delta(z - t)dz =$$

$$= \lim_{k \rightarrow +\infty} \int [h(\tau)](z)\delta(z - t)dz$$

$$\geq \lim_{k \rightarrow +\infty} \int [h(\tau)](z)p_k(z - t)dz \geq 0.$$  

Thus the function $h(\tau)$ is non-negative.

**Proof:** [Proof of proposition 6] Consider a mapping $\Theta$ satisfying $\sum_k [h(n)](k)\Theta(n)(k) = 0$, $\forall n \in \mathbb{N}$, we want to show that $\Theta$ is a minimal element. Assume that there exists $\Lambda \in \mathcal{K}(\Psi)$ such that $\Lambda \prec \Theta$. Then, from the fact that $\Lambda \in \mathcal{K}(\Psi)$ and the fact that $[h(n)](k) \geq 0$ for all $n, k \in \mathbb{N}$, we have $0 \leq \sum_k [h(n)](k)\Lambda(n)(k) \leq \sum_k [h(n)](k)\Theta(n)(k) = 0 \Longrightarrow \sum_k [h(n)](k)\Lambda(n)(k) = 0$. From the fact that $\Lambda \prec \Theta$, there exists $n$ and $j$ such that $[\Lambda(n)](j) < [\Theta(n)](j)$ and hence $[h(n)](j)[\Lambda(n)](j) < [h(n)](j)[\Theta(n)](j)$. Thus $\sum_k [h(n)](k)\Lambda(n)(k) < \sum_k [h(n)](k)\Theta(n)(k) = 0$. Contradiction. Therefore $\Theta$ is a minimal element.

Consider now a minimal element $\Theta$. We want to show that $\Theta$ satisfies $\sum_k [h(n)](k)\Theta(n)(k) = 0$, $\forall n \in \mathbb{N}$. Assume that there exists $n_0 \in \mathbb{N}$ such that $\sum_k [h(n_0)](k)\Theta(n_0)(k) = q > 0$. Consider the mapping $\Lambda : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ defined by

$$\Lambda(n) = \begin{cases} \Theta(n), & \text{if } n \neq n_0; \\ \Theta(n_0), & \text{if } n = n_0. \end{cases}$$

Then $\Lambda \prec \Theta$ and $\Lambda \in \mathcal{K}(\Psi)$ since

$$\sum_k [h(n_0)](k)\Lambda(n_0)(k) = 0 \text{ and } \sum_k [h(n)](k)\Lambda(n)(k) = 0.$$  

This contradicts the minimality of $\Theta$. Hence, $\Theta$ satisfies $\sum_k [h(n)](k)\Theta(n)(k) = 0$, $\forall n \in \mathbb{N}$. We conclude that the class of minimal elements of the kernel are exactly

$$\{ \Theta : \sum_k [h(n)](k)\Theta(n)(k) = 0, \forall n \in \mathbb{N} \}.$$  

**APPENDIX D: PROOF OF THEOREMS**

**Proof:** [Proof of theorem 1] First assume that $\Psi = \bigwedge_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta$. Then, $\Psi$ is an increasing V-system as supremum of increasing V-systems. Assume now that $\Psi$ is an increasing V-system. Consider $f \in \mathcal{U}SC(\mathbb{E})$ and $t \in T$. Let $f = \bigwedge_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta(f)$. We will show that $[\Psi(f)](x) \geq t \iff f(x) \geq t$.

Assume first that $[\Psi(f)](x) \geq t$, for some $x \in \mathbb{E}$. Consider the mapping $\Theta_{f,t}$ given by

$$\Theta_{f,t}(x) = \begin{cases} f - t, & \text{if } [\Psi(f)](x) \geq t; \\ T, & \text{otherwise}. \end{cases}$$

We have $\Theta_{f,t}(x) = f - t$. Moreover,

$$\Psi[\Theta_{f,t}(x)] = \begin{cases} \Psi(f - t), & \text{if } [\Psi(f)](x) \geq t; \\ T, & \text{otherwise.} \end{cases}$$

In particular, $\Psi[\Theta_{f,t}(x)](x) \geq 0$. So, $\Theta_{f,t} \in \mathcal{K}(\Psi)$. We have $\mathcal{E}_{\Theta_{f,t}}(f) = \bigvee_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta(f) = [\Theta_{f,t}(x) + v \leq f] \geq t$ since $t \in \{ v \in T : \Theta_{f,t}(x) + v \leq f \} \geq t$. Hence, $f(x) = \bigvee_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta(f) \geq \mathcal{E}_{\Theta_{f,t}}(f) \geq t.$
Assume now that \( f'(x) \geq t \). We have
\[
\begin{align*}
\Psi(x) & \leq \bigvee_{\Theta \in \mathcal{K}(\Psi)} E_{\Theta}(f)(x) \\
& \iff \exists \Theta \in \mathcal{K}(\Psi) : E_{\Theta}(f)(x) \geq t \\
& \iff \exists \Theta \in \mathcal{K}(\Psi) : \land_{u} \{ f(u) - [\Theta(x)](u) \} \geq t \\
& \iff \exists \Theta \in \mathcal{K}(\Psi) : f - \Theta(x) \geq t \\
& \iff \exists \Theta \in \mathcal{K}(\Psi) : (f + \Theta(x)) \leq t \\
& \iff \exists \Theta \in \mathcal{K}(\Psi) : (f + \Theta(x)) \leq t + \Psi(\Theta(x)).
\end{align*}
\]

Since \( \Theta \in \mathcal{K}(\Psi) \), we have \( \Psi(\Theta(x))(x) \geq 0 \). So, \( \Psi(f)(x) \geq t + \Psi(\Theta(x))(x) \geq t \).

Finally, we showed that
\[
\begin{align*}
\Psi(f)(x) & \geq t \iff f'(x) \geq t, \forall t \in T \\
& \iff \land_{x} \{ \Psi(f)(x) = \land_{x} \{ f'(x) \}, \forall t \in T \} \\
& \iff \Psi(f) = f'.
\end{align*}
\]

where the last equivalence follows from the bijection of the threshold sets operators [2], [7]. This establishes the proof that a function-processing system is an increasing V-system iff it is the supremum of erosions by mappings in its kernel. The dual representation of \( \Psi \) in terms of SVFP dilations is easily obtained by duality.

Proof: [Proof of theorem 2] Let \( \Psi \) be an increasing and upper-semi-continuous V-system and let \( \psi_{u} \) be its umbra processing system. From [9, Theorem 9], \( \psi_{u} \) is increasing and u.s.c. We showed in [11, Theorem 3] that an increasing u.s.c. set-processing system has a minimal element. Therefore \( \psi_{u} \) has a minimal element. Let \( \Theta_{M}^{\psi_{u}} \) be a minimal element of \( \psi_{u} \). Due to the one-to-one correspondence between \( \ker(\psi_{u}) \) and \( \mathcal{K}(\Psi) \) (see proof of Proposition 2), there exists a unique \( \Theta_{M} \in \mathcal{K}(\Psi) \) such that \( \Theta_{M}^{\psi_{u}}(x, y) = U[\Theta_{M}(x) + y], \forall (x, y) \in \mathcal{E} \times T \). Hence \( \Theta_{M} \) is a minimal element of \( \mathcal{K}(\Psi) \). For otherwise there exists \( \Lambda \in \mathcal{K}(\Psi) \) such that \( \Lambda \leq \Theta_{M} \). Let then \( \Lambda U(x, y) = U[\Lambda(x) + y], \forall (x, y) \in \mathcal{E} \times T \). From the one-to-one correspondence between \( \ker(\psi_{u}) \) and \( \mathcal{K}(\Psi) \), we deduce that \( \Lambda \) is a minimal element of \( \mathcal{K}(\Psi) \). This contradicts the fact that \( \Theta_{M}^{\psi_{u}} \) is a minimal element of \( \ker(\psi_{u}) \). Therefore, we conclude that \( \Theta_{M} \) is a minimal element of \( \mathcal{K}(\Psi) \).

Proof: [Proof of theorem 3] Let \( \Psi \) be an u.s.c. V-system and consider \( \Theta_{A} \in \mathcal{K}(\Psi) \). Then, there exists \( \Theta_{B} \in \mathcal{K}(\Psi) \) such that \( \Theta_{B} \leq \Theta_{A} \) for otherwise \( \Theta_{A} \) is a minimal element. Therefore, for every \( \Theta_{A} \in \mathcal{K}(\Psi) \), we can construct a decreasing family \( \mathcal{L} \) of \( \mathcal{K}(\Psi) \) containing \( \Theta_{A} \). From the fact that \( \mathcal{L} \) is a totally ordered subset of \( \mathcal{K}(\Psi) \) and from Hausdorff’s maximality principle [4], there exists a maximal totally ordered subset \( \mathcal{M} \) of \( \mathcal{K}(\Psi) \) containing \( \mathcal{L} \). Let \( \Theta_{M}(x) = (\land_{M}(x)) = \land_{\Theta \in \mathcal{M}} \Theta(x) \) for every \( x \in \mathcal{E} \). From [33, Lemma 4.1], there exists a sequence \( \{ \Theta_{n}(x) : n \in \mathbb{N}, \Theta_{n} \in \mathcal{M} \} \) such that \( \Theta_{n}(x) \downarrow \Theta_{M}(x) \), for every \( x \in \mathcal{E} \). From the fact that \( \Psi \) is an u.s.c. system and \( \Theta_{n}(x) \) is an u.s.c. function for every \( x \in \mathcal{E} \), we have
\[
\begin{align*}
\Theta_{n}(x) & \downarrow \Theta_{M}(x), \forall x \in \mathcal{E}.
\end{align*}
\]

We also have \( \Psi(\Theta_{n}(x)) \downarrow \land_{\Psi}(\Theta_{M}(x)), \forall x \). By the uniqueness of the limit, we have \( [\Psi(\Theta_{M}(x))(x) = \land_{n}[\Psi(\Theta_{n}(x))(x)](x)] \). Since \( \Theta_{n} \in \mathcal{K}(\Psi) \), we have \( [\Psi(\Theta_{n}(x))(x) \geq 0, \forall n \in \mathbb{N}, \forall x \in \mathcal{E} \). Hence \( [\Psi(\Theta_{M}(x))(x) \geq 0, \forall x \in \mathcal{E} \).

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