EXAMPLES 1: FOURIER SERIES

1. Find the Fourier series of each of the following functions

(i) $f(x) = 1 - x^2, \quad -1 < x < 1.$

(ii) $g(x) = |x|, \quad -\pi < x < \pi.$

(iii) $h(x) = \begin{cases} 0 \text{ if } -2 < x < 0 \\ 1 \text{ if } 0 \leq x < 2. \end{cases}$

In each case sketch the graph of the function to which the Fourier series converges over an x- range of three periods of the Fourier series.

2. Find the Fourier series for $f(x) = \frac{x^2}{4}, \; -\pi < x < \pi$. Hence deduce that

(i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots.$

(ii) $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots.$

(iii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots.$

3. Find the Fourier cosine series and the Fourier sine series for the function

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < 1 \\ 0 \text{ if } 1 \leq x < 2. \end{cases}$$

4. Find the Fourier cosine series for the function $f(x) = \sin(x), \; 0 < x < \pi$. What is the Fourier sine series for f?

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a periodic function of period $2L$ with Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}).$$

(i) Using integration by parts, show that, if f' exists and is a bounded function on $\mathbf{R}$, then there exists a constant k such that $|a_n| \leq \frac{k}{n}$ and $|b_n| \leq \frac{k}{n}$ for all $n \geq 1$.

(ii) Show that, if f'' exists and is a bounded function on $\mathbf{R}$, then the Fourier series for f is absolutely convergent for all x.

EXAMPLES 1: FOURIER SERIES – SOLUTIONS

1. (i) We must calculate the Fourier coefficients.

$a_0 = \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx = \frac{1}{2}(x-\frac{1}{3}x^3|_{-1}^{1} = \frac{2}{3}$.

$a_n = \frac{1}{1}\int_{-1}^{1}(1-x^2)\cos(n\pi x)\,dx = \int_{-1}^{1}\cos(n\pi x)\,dx - \int_{-1}^{1}x^2\cos(n\pi x)\,dx$

$= 0 - \frac{1}{n\pi}x^2\sin(n\pi x)|_{-1}^{1} + \frac{2}{n\pi}\int_{-1}^{1}x\sin(n\pi x)\,dx$

$= 0 - 0 - \frac{2}{n^2\pi^2}x\cos(n\pi x)|_{-1}^{1} + \frac{2}{n^2\pi^2}\int_{-1}^{1}\cos(n\pi x)\,dx$

$= -\frac{4}{n^2\pi^2}\cos(n\pi) + 0 = (-1)^{n+1}\frac{4}{n^2\pi^2}$.

Also $b_n = \frac{1}{1}\int_{-1}^{1}(1-x^2)\sin(n\pi x)\,dx = 0$ as integrand is odd.

Hence

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\{\cos(\pi x) - \frac{1}{4}\cos(2\pi x) + \frac{1}{9}\cos(3\pi x) - \ldots\}.$$

(ii) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{2}{2\pi}\int_0^{\pi}x\,dx = \frac{1}{\pi}\frac{1}{2}x^2|_0^{\pi} = \frac{\pi}{2}$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}x\cos(nx)\,dx$

$= \frac{2}{n\pi}x\sin(nx)|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx = 0 + \frac{2}{n^2\pi}\cos(nx)|_0^{\pi}$

$= \frac{2}{n^2\pi}[(-1)^n - 1] = \begin{cases} -\frac{4}{n^2\pi} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even.} \end{cases}$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx)\,dx = 0$ as integrand is odd.

Hence

$$g(x) = \frac{\pi}{2} - \frac{4}{\pi}\{\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \ldots\}.$$

(iii) $a_0 = \frac{1}{4}\int_{-2}^{2}h(x)\,dx = \frac{1}{4}\int_0^2 dx = \frac{1}{2}$.

$a_n = \frac{1}{2}\int_{-2}^{2}h(x)\cos(\frac{n\pi x}{2})\,dx = \frac{1}{2}\int_0^2\cos(\frac{n\pi x}{2})\,dx = \frac{1}{2}\frac{2}{n\pi}\sin(\frac{n\pi x}{2})|_0^2 = 0$.

$b_n = \frac{1}{2}\int_{-2}^{2}h(x)\sin(\frac{n\pi x}{2})\,dx = \frac{1}{2}\int_0^2\sin(\frac{n\pi x}{2})\,dx = -\frac{1}{2}\frac{2}{n\pi}\cos(\frac{n\pi x}{2})|_0^2$

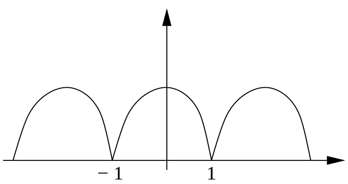
$= \frac{1}{n\pi}[1-\cos(n\pi)] = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{2}{n\pi} \text{ if } n \text{ is odd.} \end{cases}$

Hence

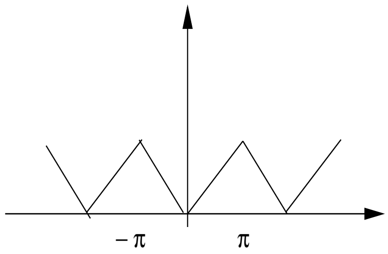
$$h(x) = \frac{1}{2} + \frac{2}{\pi}\{\sin(\frac{\pi x}{2}) + \frac{1}{3}\sin(\frac{3\pi x}{2}) + \frac{1}{5}\sin(\frac{5\pi x}{2}) + \ldots\}.$$

In (i) and (ii), if functions are extended as periodic functions, they are continuous at all points and so the Fourier series converge to the function value at all points.

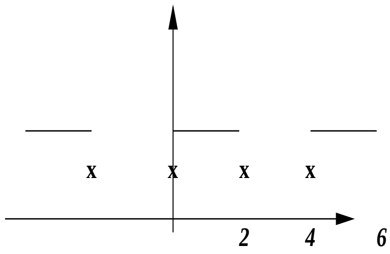
Thus the Fourier series for f converges to the following function



and the Fourier series for g converges to

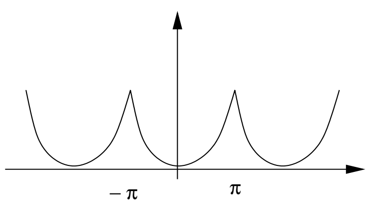


In (iii), if function is extended as a periodic function, it is discontinuous at $x = 0, \pm 2, \pm 4$; thus the Fourier series converges to $\frac{1}{2}$ at these points and converges to the value of the function at all other points.



2. Again calculating the Fourier coefficients we have
$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}\,dx = \frac{1}{2\pi}\frac{1}{4}\frac{1}{3}x^3|_{-\pi}^{\pi} = \frac{\pi^2}{12}$.
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}\cos(nx)\,dx = \frac{1}{\pi}\frac{1}{n}\frac{x^2}{4}\sin(nx)|_{-\pi}^{\pi} - \frac{1}{2\pi n}\int_{-\pi}^{\pi} x\sin(nx)\,dx$
$= 0 + \frac{1}{2\pi n^2}x\cos(nx)|_{-\pi}^{\pi} - \frac{1}{2\pi n^2}\int_{-\pi}^{\pi}\cos(nx)\,dx = \frac{1}{n^2}\cos(n\pi)$.
Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}\sin(nx)\,dx = 0$ as the integrand is odd.

Since f is continuous when extended as a periodic function with period 2π, the Fourier series for f converges to $f(x)$ for all $x \in \mathbf{R}$.



Hence
$$\frac{x^2}{4} = \frac{\pi^2}{12} - \cos(x) + \frac{1}{2^2}\cos(2x) - \frac{1}{3^2}\cos(3x) + \frac{1}{4^2}\cos(4x) - \ldots$$
Setting $x = \pi$, we have
$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.$$

i.e.,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{4}-\frac{\pi^2}{12}=\frac{\pi^2}{6} \qquad (1)$$

Setting $x=0$, we have

$$0=\frac{\pi^2}{12}-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots$$

i.e.,

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\ldots=\frac{\pi^2}{12} \qquad (2)$$

$(1)+(2)$ gives

$$2(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}.$$

Hence

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

3. The Fourier cosine series of f is given by $a_0+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{2})$ where

$$a_0=\frac{1}{4}\int_{-2}^{2}f(x)\,dx=\frac{1}{2}\int_{0}^{2}f(x)\,dx=\frac{1}{2}\int_{0}^{1}dx=\frac{1}{2}$$

and

$$\begin{aligned} a_n &= \tfrac{1}{2}\textstyle\int_{-2}^{2}f(x)\cos(\frac{n\pi x}{2})\,dx=\int_0^1\cos(\frac{n\pi x}{2})\,dx=\frac{2}{n\pi}\sin(\frac{n\pi x}{2})|_0^1 \\ &= \tfrac{2}{n\pi}\sin(\tfrac{n\pi}{2})=\begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{2}{n\pi} \text{ if } n=4k+1 \\ -\frac{2}{n\pi} \text{ if } n=4k+3. \end{cases} \end{aligned}$$

Hence

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\{\cos(\frac{\pi x}{2})-\frac{1}{3}\cos(\frac{3\pi x}{2})+\frac{1}{5}\cos(\frac{5\pi x}{2})-\ldots\}.$$

The Fourier sine series of f is given by $\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{2})$ where

$$\begin{aligned} b_n &= \tfrac{1}{2}\textstyle\int_{-2}^{2}f(x)\sin(\frac{n\pi x}{2})\,dx=\int_0^1\sin(\frac{n\pi x}{2})\,dx=-\frac{2}{n\pi}\cos(\frac{n\pi x}{2})|_0^1 \\ &= \tfrac{2}{n\pi}(1-\cos(\tfrac{n\pi}{2}))=\begin{cases} 2/n\pi \text{ if } n \text{ is odd} \\ 0 \text{ if } n=4k \\ 4/n\pi \text{ if } n=4k+2. \end{cases} \end{aligned}$$

Hence

$$f(x)=\frac{2}{\pi}\{\sin(\frac{\pi x}{2})+\frac{2}{2}\sin(\frac{2\pi x}{2})+\frac{1}{3}\sin(\frac{3\pi x}{2})+\sin(\frac{5\pi x}{2})+\frac{2}{6}\sin(\frac{6\pi x}{2})+\ldots\}.$$

4. The Fourier cosine series of f is given by $a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = (\frac{1}{\pi}).(-\cos(x))|_0^{\pi} = \frac{2}{\pi}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx.$$

If $n = 1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\,dx = 0$.
In $n \neq 1$, $a_n = \frac{1}{\pi}\int_0^{\pi}\{\sin[(n+1)x] + \sin[(1-n)x]\}\,dx$
$= \frac{1}{\pi}\{-\frac{1}{n+1}\cos[(n+1)x] + \frac{1}{n-1}\cos[(n-1)x]\}\,|_0^{\pi}$
$= \frac{1}{\pi}\{\frac{1}{n+1}(1-\cos[(n+1)\pi]) + \frac{1}{n-1}(\cos[(n-1)\pi] - 1)\}$.
Thus $a_n = 0$ if n is odd and $a_n = \frac{2}{\pi}(\frac{1}{n+1} - \frac{1}{n-1}) = -\frac{4}{\pi(n^2-1)}$ if n is even.
Hence

$$\sin(x) = \frac{2}{\pi} - \frac{4}{\pi}\{\frac{1}{3}\cos(2x) + \frac{1}{15}\cos(4x) + \frac{1}{35}\cos(6x) + \ldots\}.$$

The Fourier sine series is simply $f(x) = \sin(x)$

5(i) Since f' is a bounded function on $\mathbf{R}$, there exists a constant $c > 0$ such that $|f'(x)| \leq c$ for all $x \in \mathbf{R}$. Now

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx = \frac{1}{n\pi} f(x)\sin(\frac{n\pi x}{L})|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin(\frac{n\pi x}{L})\,dx.$$

As $x \to f(x)\sin(\frac{n\pi x}{L})$ has period $2L$ (or alternatively as $\sin(\pm n\pi) = 0$), $f(x)\sin(\frac{n\pi x}{L}|_{-L}^{L} = 0$.
Hence $a_n = -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin(\frac{n\pi x}{L})\,dx$ and so

$$|a_n| \leq \frac{2L}{n\pi}\max_{-L\leq x\leq L}|f'(x)\sin(\frac{n\pi x}{L})| \leq \frac{2Lc}{n\pi} = \frac{k}{n}.$$

A similar argument shows that $|b_n| \leq \frac{k}{n}$.
(ii) Since f'' is a bounded function on $\mathbf{R}$, there exists a constant c_1 such that $|f''(x)| \leq c_1$ for all $x \in \mathbf{R}$. By above $a_n = -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin(\frac{n\pi x}{L})\,dx$
$= \frac{L}{n^2\pi^2} f'(x)\cos(\frac{n\pi x}{L})|_{-L}^{L} - \frac{L}{n^2\pi^2}\int_{-L}^{L} f''(x)\cos(\frac{n\pi x}{L})\,dx$
$= -\frac{L}{n^2\pi^2}\int_{-L}^{L} f''(x)\cos(\frac{n\pi x}{L})\,dx$ (as $x \to f'(x)\cos(\frac{n\pi x}{L})$ has period $2L$).
Hence

$$|a_n| \leq \frac{2L^2}{n^2\pi^2}\max_{-L\leq x\leq L}|f''(x)\cos(\frac{n\pi x}{L})| \leq \frac{2L^2 c_1}{n^2\pi^2} = \frac{k_1}{n^2}.$$

A similar argument shows that $|b_n| \leq \frac{k_1}{n^2}$.
Thus

$$|a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})| \leq |a_n| + |b_n| \leq \frac{2k_1}{n^2}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, it follows from the comparison test that $\sum_{n=0}^{\infty}|a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})|$ converges.
Hence $\sum_{n=0}^{\infty}(a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L}))$ converges absolutely.

Fourier series for output voltages of inverter waveforms.

The Fourier series for a periodic function $v_o(\omega t)$ can be expressed as

$$v_o(\omega t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

For an odd quarter-wave symmetry waveform,

$$a_o = 0 \quad a_n = 0$$

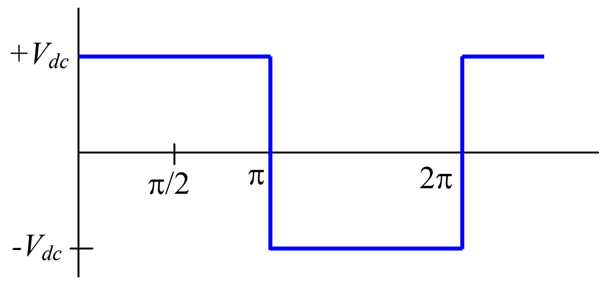
and

$$b_n = \begin{cases} \dfrac{4}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} v_o \sin(n\omega t)d(\omega t) & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

Therefore, $v_o(\omega t)$ can be written as

$$v_o(\omega t) = \sum_{n=odd}^{\infty} b_n \sin(n\omega t)$$

(i) Square-wave



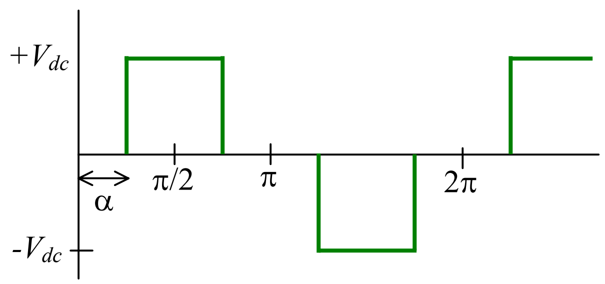
$$b_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} v_o \sin(n\omega t)d(\omega t)$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} V_{dc} \sin(n\omega t)d(\omega t)$$

$$= \frac{4V_{dc}}{n\pi}\left[-con(n\omega t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{4V_{dc}}{n\pi}$$

(ii) Quasi square-wave



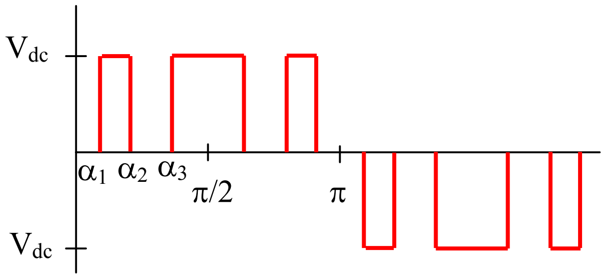
$$b_n = \frac{4}{\pi}\int_\alpha^{\frac{\pi}{2}} v_o \sin(n\omega t)d(\omega t)$$

$$= \frac{4}{\pi}\int_\alpha^{\frac{\pi}{2}} V_{dc} \sin(n\omega t)d(\omega t)$$

$$= \frac{4V_{dc}}{n\pi}\left[-con(n\omega t\right]_\alpha^{\frac{\pi}{2}}$$

$$= \frac{4V_{dc}}{n\pi}\cos(n\alpha)$$

(iii) Notched waveform (Harmonics Elimination PWM)



$$b_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} v_o \sin(n\omega t)d(\omega t)$$

$$= \frac{4}{\pi}\left[\int_{\alpha_1}^{\alpha_2} V_{dc}\sin(n\omega t)d(\omega t) + \int_{\alpha_3}^{\frac{\pi}{2}} V_{dc}\sin(n\omega t)d(\omega t)\right]$$

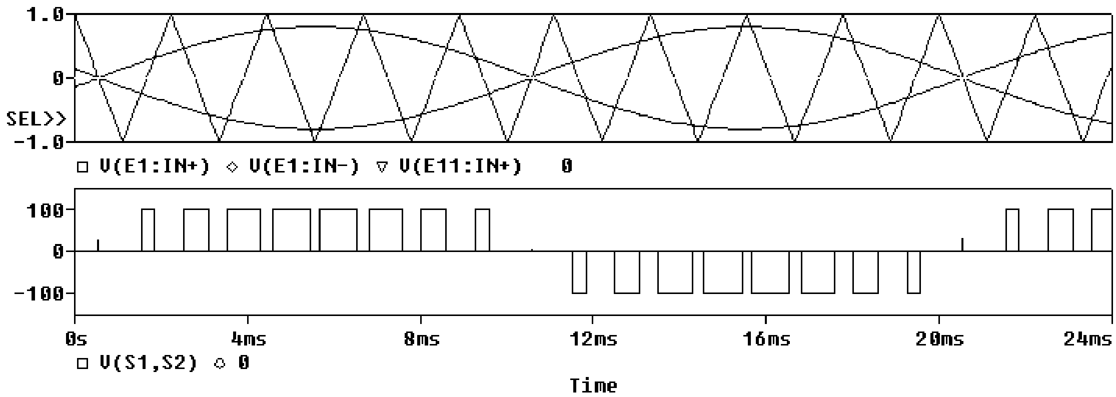
$$= \frac{4V_{dc}}{n\pi}\left\{[-con(n\omega t]_{\alpha_1}^{\alpha_2} + [-con(n\omega t]_{\alpha_3}^{\frac{\pi}{2}}\right\}$$

$$= \frac{4V_{dc}}{n\pi}\cos(n\omega t)\Big|_{\alpha_2,\frac{\pi}{2}}^{\alpha_1,\alpha_3}$$

$$= \frac{4V_{dc}}{n\pi}\left[\cos(n\alpha_1) + \cos(n\alpha_3) - \cos(n\alpha_2) - \cos(n\frac{\pi}{2})\right]$$

$$= \frac{4V_{dc}}{n\pi}[\cos(n\alpha_1) + \cos(n\alpha_3) - \cos(n\alpha_2)]$$

(iv) Sinusoidal PWM (unipolar and bipolar)



Unipolar SPWM waveform as an example

$$v_o(\omega t) = M_a V_{dc}\sin(\omega t) + \frac{4V_{dc}}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left[\frac{n\pi}{2} + (\frac{k}{2})\cos(\omega t)\right]}{n}$$
$$= M_a V_{dc}\sin(\omega t) + \text{Bessel Function for harmonic terms}$$

The tables are required to resolve for the Bessel function for harmonic terms. The harmonics in the inverter output appear as sidebands, centered around the switching frequency, that is, around mf, $2m_f$, $3m_f$ and so on. This general pattern hold true for all values of ma in the range 0 – 1 and m_f>9. The unipolar SPWM switching scheme has the advantage of "effectively" doubling the switching frequency as far as the output harmonics are concerned, compared to the bipolar SPWM switching scheme. Because of that the harmonics in the inverter output of unipolar SPWM are centered around $2m_f$, $4m_f$, $6m_f$ and so on.

TABLE 8.3 NORMALIZED FOURIER COEFFICIENTS V_n/V_{dc} FOR BIPOLAR SPWM

	$M_a = 1$	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
n = 1	1.00	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
$n = m_f$	0.60	0.71	0.82	0.92	1.01	1.08	1.15	1.20	1.24	1.27
$n = m_f \pm 1$	0.32	0.27	0.22	0.17	0.13	0.09	0.06	0.03	0.02	0.00

Table 8.3 shows the first harmonic frequencies in the output spectrum at and around m_f for the bipolar SPWM switching scheme. The harmonics at and around $2m_f$, $3m_f$, $4m_f$ and so on are not indicated.

TABLE 8.5 NORMALIZED FOURIER COEFFICIENTS V_n/V_{dc} FOR UNIPOLAR SPWM

	$M_a = 1$	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
n = 1	1.00	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
$n = 2m_f \pm 1$	0.18	0.25	0.31	035	0.37	0.36	0.33	0.27	0.19	0.10
$n = 2m_f \pm 3$	0.21	0.18	0.14	0.10	0.07	0.04	0.02	0.01	0.00	0.00

Table 8.5 shows the first harmonic frequencies in the output spectrum at and around $2m_f$ for the uipolar SPWM switching scheme. The harmonics at and around $4m_f$, $6m_f$, $8m_f$ and so on are not indicated.

Table 8.3 and 8.5 can be used to predict the THD for the ouput current of the inverter connected to RL load. Higher order harmonics are assumed to contribute little power and effect, so they can be neglected. To evaluate the THD for the output voltage of the inverter, higher order harmonics should be taken into account.