TABLE IV
DELAY OF FA-BASED AU_m'S ROM-BASED AU_m'S, AND ROM-BASED GENERAL MULTIPLIERS

	Delay (Δ)				
	FA-based AU_m's			ROM-based AU_m's	ROM-based General Multipliers
n	FA Arrays	ROMs	Total	Total	Total
3	22	6	28	18	36
4	42	9	51	36	72
5	48	11	59	55	110
6	62	12	74	72	144
7	106	12	118	84	168

TABLE V
TIME-COMPLEXITY PRODUCT OF FA-BASED AU_m'S, ROM-BASED AU_m'S, AND ROM-BASED GENERAL MULTIPLIERS

	Time-Complexity Product		
n	FA-based AU_m	ROM-based AU_m	ROM-based General Multiplier
3	11284	3402	13608
4	52020	23040	92160
5	91863	81675	326700
6	181300	306720	1226880
7	467896	775572	3102288

performance of FA-based AU_m's was analyzed and compared to that of ROM-based designs in terms of their hardware complexity, execution time, and time-complexity product.

It was found that the hardware complexity and the time-complexity product of FA-based AU_m's become less than those of ROM-based AU_m's for large moduli word lengths. The delays of FA-based AU_m's are longer than those of ROM-based AU_m's. There is a tradeoff between execution times and hardware complexity. It should be noted, though, that the FA-based AU_m's perform general multiplication, while the simple version of ROM-based AU_m's uses fixed multipliers. This makes the FA-based units a better choice for most DSP applications, such as convolution, correlation, digital filtering, discrete Fourier transforms (DFT), and vector/matrix arithmetic operations, which require general multiplication. For all moduli, the hardware complexity, delay, and time-complexity product of FA-based AU_m's are less than those of ROM-based general multipliers.

Since the proposed AU_m's use full adders as their basic units, they present good modularity and regularity. The lower hardware complexity of the proposed units makes their VLSI implementation easier for fabrication and more cost-effective than ROM-based designs.

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On the Hysteresis and Robustness of Hopfield Neural Networks

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***Abstract*— The effect of noise degradation on the Hopfield neural network is studied. The notion of a *hysteresis network* is defined. A noisy Hopfield neural network is subsequently proven to be a hysteresis network. The effect of the hysteresis phenomenon on the robustness of the Hopfield neural network to noise degradation is then investigated. An optimal Hopfield neural network is defined as the Hopfield neural network which minimizes an upper-bound on the probability of error. The minimal robustness indicator of a Hopfield neural network is defined. The upper bound on the probability of error of a noisy Hopfield neural network is derived in terms of the minimal robustness indicator. We finally prove that an optimal Hopfield neural network is obtained when the minimal robustness indicator is maximized.**

I. INTRODUCTION

Over the past decade we have witnessed the emergence of a tremendous effort in the investigation of neural networks [1]–[9]. Hopfield's proposed neural network has been one of the main factors in the rekindled interest in neural networks [1], [2]. An important reason for the ubiquity of neural networks is attributed to the success of the Hopfield network in various applications (e.g., the solution of several optimization problems [6], [7]).

Neural networks' origin is in neurophysiological models [10]–[12]. In [13], [14] psychophysical experiments have been conducted which demonstrate the existence of hysteresis in human neurophysiological perception. It is intriguing to determine whether hysteresis is present in neural networks as well. Furthermore, we are interested in demonstrating the possible effects of hysteresis on the robustness of the neural network to noise degradation.

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A most promising future niche of neural networks is in high-speed massively parallel integrated circuits [9]. Recent efforts have concentrated on the implementation of neural networks on integrated circuits [15]–[17]. The practical utility of the neural network in hardware implementation, however, critically depends upon the robustness of the neural network to noise degradation.

In this paper, we study the effect of noise degradation on the Hopfield neural network. In Section II, a dynamical analysis of the effect of noise degradation on neural networks is proposed. The dynamical analysis is based on the hysteresis phenomenon. The *Hopfield neural network* is first introduced. The *noisy Hopfield neural network* is also presented. This model represents the cumulative effect of noise degradation on the implementation of the Hopfield neural network, e.g., roundoff error, thermal noise, etc. We then define the notion of a *hysteresis network.* Subsequently, we prove that a noisy Hopfield neural network is a hysteresis network. In Section III, the robustness of the Hopfield neural network to the effect of noise degradation is investigated. An optimal Hopfield neural network is defined as the Hopfield neural network which minimizes an upper bound on the probability of error. The *minimal robustness indicator* of a Hopfield neural network is defined. The upper bound on the probability of error of a noisy Hopfield neural network is derived in terms of the minimal robustness indicator. We prove that an optimal Hopfield neural network is obtained when the minimal robustness indicator is maximized. Several computer simulations are used to illustrate the effect of noise degradation on the Hopfield neural network predicted by our results. Finally, in Section IV, a brief discussion and summary is presented.

II. Hysteresis

In this section, we propose a dynamical analysis of the effect of noise degradation on neural networks. The dynamic analysis is based on the hysteresis phenomenon. We prove that a noisy Hopfield neural network is a hysteresis network.

Consider the *limiting function* $h(\cdot)$ given by

$$h(x) = \begin{cases} +1, & x \geq 0; \\ -1, & x < 0. \end{cases} \tag{1}$$

In the following we use M and N to denote the number of iterations and number of nodes, respectively.

Let us consider an *initial state* φ_i^0, for $i = 0, 1, \cdots, N-1$. Let us also consider a *network matrix* c_{ij}, for $i, j = 0, 1, \cdots, N-1$. The *Hopfield neural network* is defined by

$$\varphi_i^{k+1} = h\left(\sum_{j=0}^{N-1} c_{ij}\varphi_j^k\right), \tag{2}$$

for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$ [9]. The resulting sequence φ_i^{k+1}, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, is known as a *state process.*

As noted earlier, the Hopfield neural network has been demonstrated to provide a very effective approach to several problems [6], [7]. Nonetheless, the performance of the Hopfield neural network is hampered by the persistence of numerous imperfections, e.g., spurious states [1], [2]. The performance of the Hopfield neural network is effected in practice by various implementation considerations, e.g., roundoff error, thermal noise, etc. In the following we model the cumulative effect of noise degradation on the implementation of the Hopfield neural network.

Let us consider a *noise process* η_i^{k+1}, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$. Let us also consider an initial state $\tilde{\varphi}_i^0 = \varphi_i^0$, for $i = 0, 1, \cdots, N-1$. The *noisy Hopfield neural network* is defined by

$$\tilde{\varphi}_i^{k+1} = h\left(\sum_{j=0}^{N-1} c_{ij}\tilde{\varphi}_j^k + \eta_i^{k+1}\right), \tag{3}$$

for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$. The resulting sequence $\tilde{\varphi}_i^{k+1}$, for $k = 0, 1, \cdots, N-1$ and for $i = 0, 1, \cdots, N-1$, is known as a *noisy state process.*

In the following definition we introduce the notion of a hysteresis network.

Definition 1: A mapping $H(\cdot)$ from a process η_i^{k+1} to a process $\tilde{\varphi}_i^{k+1}$, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, is a *hysteresis network* if there exists a sequence η_i^{k+1}, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, such that, if $\eta_i^{k_1} = \eta_i^{k_2}$, for all $i = 0, 1, \cdots, N-1$, then $\tilde{\varphi}_i^{k_1} \neq \tilde{\varphi}_i^{k_2}$, for some $i = 0, 1, \cdots, N-1$, and for some $k_1 \neq k_2$.

In the following proposition we verify the presence of the hysteresis phenomenon in the noisy Hopfield neural network.

Proposition 1: A noisy Hopfield neural network is a hystersis network.

Proof: Let Λ_i be given by

$$\Lambda_i = \sum_{j=0}^{N-1} |c_{ij}|,$$

for $i = 0, 1, \cdots, N-1$. Let $\eta_i^1 < -\Lambda_i$, for $i = 0, 1, \cdots, N-1$. From (1) and (3) we observe that $\tilde{\varphi}_i^1 = -1$, for $i = 0, 1, \cdots, N-1$. Setting $\eta_i^2 = 0$, for $i = 0, 1, \cdots, N-1$, we have $\tilde{\varphi}_i^2 = h(-\sum_{j=0}^{N-1} c_{ij})$, for $i = 0, 1, \cdots, N-1$. We now let $\eta_i^3 > \Lambda_i$, for $i = 0, 1, \cdots, N-1$. From (1) and (3) we observe that $\tilde{\varphi}_i^3 = +1$, for $i = 0, 1, \cdots, N-1$. Setting $\eta_i^4 = 0$, for $i = 0, 1, \cdots, N-1$, we have $\tilde{\varphi}_i^4 = h(\sum_{j=0}^{N-1} c_{ij})$, for $i = 0, 1, \cdots, N-1$. Finally, we observe that $\eta_i^2 = \eta_i^4 = 0$ and $\tilde{\varphi}_i^2 = -\tilde{\varphi}_i^4$, for $i = 0, 1, \cdots, N-1$. This completes the proof □.

From Proposition 1 we observe that, given a noisy Hopfield neural network, there exists a noise process η_i^{k+1}, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, such that, if $\eta_i^{k_1} = \eta_i^{k_2}$, for all $i = 0, 1, \cdots, N-1$, then $\tilde{\varphi}_i^{k_1} \neq \tilde{\varphi}_i^{k_2}$, for some $i = 0, 1, \cdots, N-1$, and for some $k_1 \neq k_2$ (see (3)).

Once the presence of hysteresis in noisy Hopfield neural networks has been established our interest is focused at the investigation of the possible ramifications of this phenomenon. In particular, we are interested in demonstrating the possible effects of hysteresis on the robustness of the Hopfield neural network to noise degradation. This is the subject of the next section.

III. Robustness

In this section, we investigate the robustness of the Hopfield neural network to the effect of noise degradation.

We restrict ourselves to independent and identically distributed random noise process $\{\eta_i^{k+1}: k = 0, 1, \cdots, M-1, i = 0, 1, \cdots, N-1\}$; i.e., the collection of probability density functions $\{p_{\eta_i^{k+1}}(\eta_i^{k+1}): k = 0, 1, \cdots, M-1, i = 0, 1, \cdots, N-1\}$ is given by

$$p_{\eta_i^{k+1}}(\eta_i^{k+1}) = p(\eta), \tag{4}$$

for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$.

We define an *optimal Hopfield neural network* $\hat{c}_{ij}$ for $i, j = 0, 1, \cdots, N-1$, as the Hopfield neural network which minimizes the probability of error $P_E[c_{ij}]$; i.e., the optimal Hopfield neural network is given by

$$\hat{c}_{ij} = \text{argmin}_{c_{ij}} P_E[c_{ij}], \tag{5a}$$

where

$$P_E[c_{ij}] = 1 - P(\tilde{\varphi}_i^M = \varphi_i^M, \quad i = 0, 1, \cdots, N-1). \tag{5b}$$

It is important to observe that the (noisy) Hopfield neural network is completely characterized by the network matrix c_{ij}, for $i, j = 0, 1, \cdots, N-1$ (see (2) and (3)).

An exact derivation of the probability of error $P_E[c_{ij}]$ is a formidable task. Consequently, the determination of the optimal Hopfield neural network $\hat{c}_{ij}$, for $i, j = 0, 1, \cdots, N-1$, given by (5), remains unsolved. This problem, however, can be circumvented by considering an alternative definition of the optimal Hopfield neural network.

We shall now define an *optimal Hopfield neural network* $\hat{c}_{ij}$ for $i, j = 0, 1, \cdots, N-1$, as the Hopfield neural network which minimizes an upper bound on the probability of error $P_E^u[c_{ij}]$; i.e., the optimal Hopfield neural network is given by

$$\hat{c}_{ij} = \text{argmin}_{c_{ij}} P_E^u[c_{ij}], \tag{6a}$$

where

$$P_E^u[c_{ij}] \geq P_E[c_{ij}]. \tag{6b}$$

In the following definition we introduce a collection of robustness indicators.

Definition 2: The *minimal robustness indicator* λ is given by

$$\lambda = \min\{\lambda_i\colon\ i = 0, 1, \cdots, N-1\}; \tag{7a}$$

whereas the *ith-order robustness indicator* λ_i is given by

$$\lambda_i = \min\left\{\left|\sum_{j=0}^{N-1} c_{ij}x_j\right| \ x_j \in \{-1, +1\}\right\}, \tag{7b}$$

for $i = 0, 1, \cdots, N-1$.

In the following lemma we derives sufficient conditions for the Hopfield neural network to be robust to noise degradation.

Lemma 1: If $|\eta_i^{k+1}| < \lambda_i$, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, then $\tilde{\varphi}_i^{k+1} = \varphi_i^{k+1}$, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$.

Proof: From (1), (3), and (7) we observe that

$$\tilde{\varphi}_i^{k+1} = h\left(\sum_{j=0}^{N-1} c_{ij}\tilde{\varphi}_j^k + \eta_i^{k+1}\right) = h\left(\sum_{j=0}^{N-1} c_{ij}\tilde{\varphi}_j^k\right),$$

for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$. Therefore, $\tilde{\varphi}_i^{k+1} = \varphi_i^{k+1}$, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$ (see also (2)).

This completes the proof. □

From Lemma 1 we observe that, if the noise process η_i^{k+1} is bounded by the ith order robustness indicator λ_i, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, then the noisy Hopfield neural network is uneffected by the degradation noise (i.e., the Hopfield neural network is equivalent to the noisy Hopfield neural network). Notice also that, if the noise process η_i^{k+1} is bounded by the minimal robustness indicator λ, for $k = 0, 1, \cdots, M-1$ and for $i = 0, 1, \cdots, N-1$, then the noisy Hopfield neural network is uneffected by the degradation noise as well.

In the following theorem we derive an upper bound on the probability of error of a noisy Hopfield neural network.

Theorem 1: The optimal Hopfield neural network is given by

$$\hat{c}_{ij} = \text{argmin}_{c_{ij}} P_E^u[c_{ij}], \tag{8a}$$

where

$$P_E^u[c_{ij}] = 1 - \left[\int_{-\lambda}^{\lambda} p(\eta)\, d\eta\right]^{NM} \tag{8b}$$

Proof: From (2)–(7), and Lemma 1 we observe that

$$\begin{aligned} P_E[c_{ij}] &= 1 - P(\tilde{\varphi}_i^M = \varphi_i^M,\ i = 0, 1, \cdots, N-1) \\ &\leq 1 - P(\tilde{\varphi}_i^{k+1} = \varphi_i^{k+1},\ i = 0, 1, \cdots, N-1, \\ &\qquad\qquad k = 0, 1, \cdots, M-1) \\ &= 1 - \prod_{k=0}^{M-1}\prod_{i=0}^{N-1} P(\tilde{\varphi}_i^{k+1} = \varphi_i^{k+1}) \\ &\leq 1 - \prod_{k=0}^{M-1}\prod_{i=0}^{N-1} \int_{-\lambda_i}^{\lambda_i} p(\eta)\, d\eta \\ &= 1 - \left[\prod_{i=0}^{N-1} \int_{-\lambda_i}^{\lambda_i} p(\eta)\, d\eta\right]^M \\ &\leq 1 - \left[\prod_{i=0}^{N-1} \int_{-\lambda}^{\lambda} p(\eta)\, d\eta\right]^M \\ &= 1 - \left[\int_{-\lambda}^{\lambda} p(\eta)\, d\eta\right]^{NM} \end{aligned}$$

This completes the proof. □

From Theorem 1 we observe that the upper bound on the probability of error of a noisy Hopfield neural network $P_E^u[c_{ij}]$ is derived in terms of the minimum robustness indicator λ.

In the following corollary we derive a criterion for the determination of the optimal Hopfield neural network.

Corollary 1: The optimal Hopfield neural network is given by

$$\hat{c}_{ij} = \text{argmax}_{c_{ij}} \lambda. \tag{9}$$

Proof: The proof is obtained directly from (8).

This completes the proof. □

From Corollary 1 we observe that an optimal Hopfield neural network is obtained when the minimal robustness indicator λ is maximized.

In the following example several computer simulations are used to illustrate the effect of noise degradation on the Hopfield neural network predicted by Corollary 1.

Example 1: Let us consider a Hopfield neural network given by

$$c_{ij} = c \cdot \tilde{c}_{ij}, \tag{10}$$

where $0 \leq c \leq \mu$. A restriction μ on the value of the scalar multiple c reflects possible practical limitations (e.g., implementation considerations, increased multiplicative noise, etc.). From (9) we observe that the optimal Hopfield neural network is given by $\hat{c} = \mu$ (i.e., $\hat{c}_{ij} = \mu \cdot \tilde{c}_{ij}$).

Fig. 1 illustrates the results of this example. Consider the Hopfield neural network, given by (10), where the coefficients $\tilde{c}_{ij} = h_{ij}$ and the coefficients h_{ij} denote the Hopfield coefficients [9]. Let the probability density function $p(\eta)$ correspond to a Gaussian distribution $N(0, \sigma^2)$ [18]. Finally, the results obtained with $N = 100$ and $M = 100$, where 5 exemplars and 100 patterns were used, are depicted in Fig. 1(a); whereas, the results obtained with $N = 64$ and $M = 100$, where 4 exemplars and 100 patterns were used, are depicted in Fig. 1(b). Observe that the probability of error computed accounts for errors generated by two distinct sources. The first source is obviously the external noise process introduced in the noisy Hopfield neural network. The second source, however, is due to imperfections of the original Hopfield neural network; e.g., spurious states. This is most easily demonstrated by the fact that the probability of error corresponding to no external noise ($\sigma = 0$) does not vanish (see Fig. 1). It is important to note that some of the imperfections of the original Hopfield neural network could be interpreted as a consequence of noise degradation (i.e., roundoff

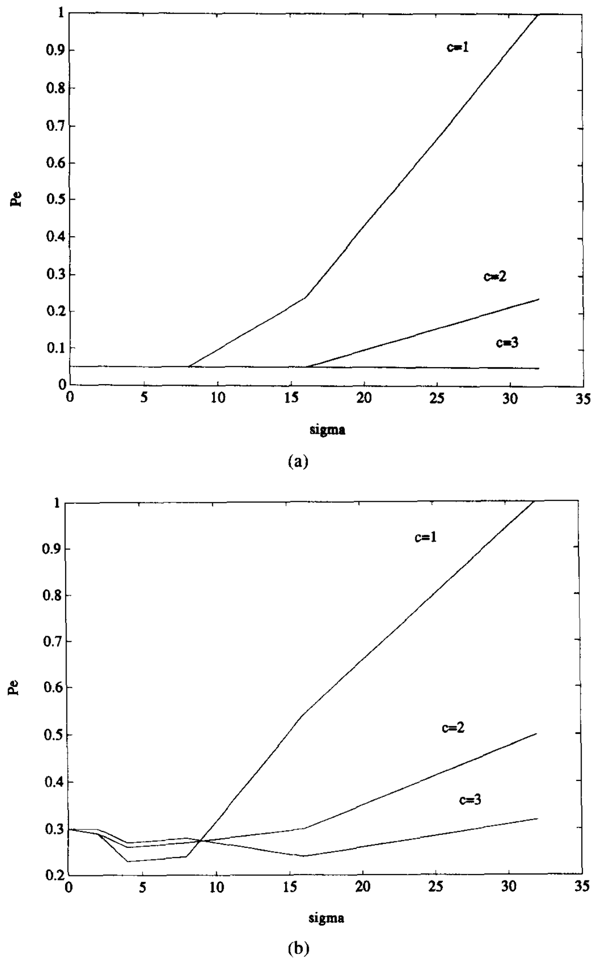

Fig. 1. Consider the noisy Hopfield neural network, given by (10), where the coefficients $\tilde{c}_{ij} = h_{ij}$ and the coefficients h_{ij} denote the Hopfield coefficients [1]. Let the probability density function $p(\eta)$ correspond to a Gaussian distribution $N(0, \sigma^2)$ [18]. (a) Probability of error obtained with $N = 100$ and $M = 100$, where 5 examplers and 100 patterns were used. (b) Probability of error obtained with $N = 64$ and $M = 100$, where 4 examplers and 100 patterns were used.

error). As expected the probability of error observed in Fig. 1(a) decreases as the value of c increases. An interesting phenomenon is observed in Fig. 1(b). Although the probability of error decreases as the values of c increases when the noise level is high, as expected; the probability of error increases as the value of c increases when the noise level is mild. Moreover, the probability of error is minimized at a value of $\sigma \neq 0$. Thus, the presence of mild noise levels improves the performance of the Hopfield neural network. A theoretical justification of this phenomenon and the determination of the optimal noise level, however, remain yet unresolved.

IV. Summary

In this paper we have studied the effect of noise degradation on the Hopfield neural network. We proved that a noisy Hopfield neural network network is a hysteresis network. We proved that a noisy Hopfield neural network is a hysteresis network. The effect of the hysteresis phenomenon on the robustness of the Hopfield neural network to noise degradation was subsequently investigated. An optimal Hopfield neural network is defined as the Hopfield neural network which minimizes an upper bound on the probability of error. We proposed the minimal robustness indicator of a Hopfield neural network. The upper bound on the probability of error of a noisy Hopfield neural network has been derived in terms of the minimal robustness indicator. We finally proved that an optimal Hopfield neural network is obtained when the minimal robustness indicator is maximized. The various results obtained in this paper had been verified by computer simulations. An interesting phenomenon, however, has been observed in the computer simulations whereby the probability of error is minimized at a mild (nonzero) noise level. Thus, the presence of mild noise levels improves the performance of the Hopfield neural network. The determination of the optimal noise level remains an open problem. Although the presence of hysteresis in noisy Hopfield neural networks can be extended to any limiting function (see (1)), the criterion for the determination of the optimal Hopfield neural network does not. Hence, alternative parameters must be obtained in the future in order to characterize the robustness of other neural networks.

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