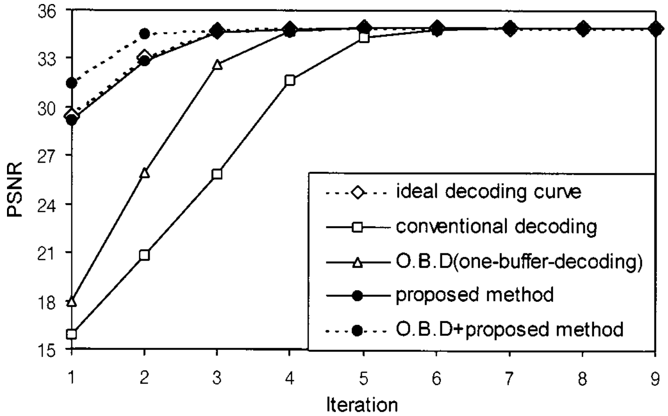
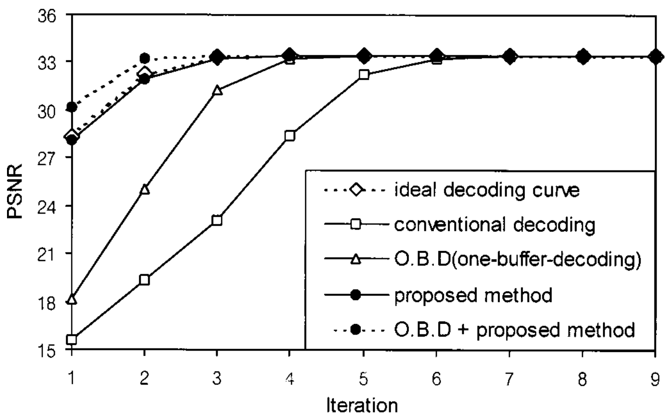


(a) Lena



(b) boat

Fig. 5. Convergence graph in decoding (a) Lena and (b) boat images. Pepper image is used as the initial image for one-buffer-decoding [5] and conventional decoding. The output image of the first stage is used as the initial image in the proposed method.

Besides, the proposed method can be generally applied to the various compression approaches because no constraints are imposed on the encoding procedure.

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REFERENCES

[1] A. E. Jacquin, "Image coding based on a fractal theory of iterated contractive image transformations," *IEEE Trans. Image Processing*, vol. 1, pp. 18–30, Jan. 1992.
[2] H. O. Peitgen, H. Jurgen, and D. Saupe, *Chaos and Fractals—New Frontiers of Science*, New York: Springer-Verlag, 1992.
[3] Z. Baharav, D. Malah, and E. Karnin, "Hierarchical interpretation of fractal image coding and its application to fast decoding," in *Proc. Digital Signal Processing*, July 1993, pp. 190–195.
[4] Y. Fisher and S. Menlove, "Fractal encoding with HV partitions," in *Fractal Image Compression—Theory and Application*, New York: Springer-Verlag, 1994.
[5] G. E. Oien and S. Lepsoy, "Fractal-based image coding with fast decoder convergence," *Signal Processing*, vol. 40, pp. 105–117, 1994.
[6] H. S. Kang and S. D. Kim, "A fractal decoding algorithm for fast convergence," *Opt. Eng.*, vol. 35, pp. 3191–3198, Nov. 1996.
[7] Y. H. Moon, K. R. Baek, Y. S. Kim, and J. H. Kim, "A fast fractal decoding algorithm with convergence criteria," *Opt. Eng.*, vol. 36, pp. 1992–1999, July 1997.
[8] L. Thomas and F. Deravi, "Region-based fractal image compression using heuristic search," *IEEE Trans. Image Processing*, vol. 4, pp. 832–838, June 1995.
[9] D. Saupe and M. Ruhl, "Evolutionary fractal image compression," in *Proc. IEEE Int. Conf. Image Processing*, 1996, pp. 129–132.
[10] E. Kreyszig, *Introductory Functional Analysis with Applications*, New York: Wiley, 1996.

On the Relation of Order-Statistics Filters and Template Matching: Optimal Morphological Pattern Recognition

Dan Schonfeld

***Abstract*—In this paper, we investigate methods for optimal morphological pattern recognition. The task of optimal pattern recognition is posed as a solution to a hypothesis testing problem. A minimum probability of error decision rule—maximum *a posteriori filter*—is sought. The classical solution to the minimum probability of error hypothesis testing problem, in the presence of independent and identically distributed noise degradation, is provided by template matching (TM). A modification of this task, seeking a solution to the minimum probability of error hypothesis testing problem, in the presence of composite (mixed) independent and identically distributed noise degradation, is demonstrated to be given by weighted composite template matching (WCTM). As a consequence of our investigation, the relationship of the order-statistics filter (OSF) and TM—in both the standard as well as the weighted and composite implementations—is established. This relationship is based on the thresholded cross-correlation representation of the OSF. The optimal order and weights of the OSF for pattern recognition are subsequently derived. An additional outcome of this representation is a fast method for the implementation of the OSF.**

***Index Terms*—Morphological filters, order-statistics filters, pattern recognition, template matching.**

I. INTRODUCTION

Morphological filters have been used extensively in various signal and image processing applications over the past decade [2], [5]. Their influence has been particularly significant in image enhancement and restoration as well as image compression and communication. A similar impact of morphological filters on pattern recognition, however, has not yet been thoroughly explored despite various attempts.

Our goal is the investigation of methods for the implementation of morphological filters in template matching: the determination of the location of the degraded version of a shifted template. A standard approach to this problem is based on template matching (TM)—the maxima of the cross-correlation of the image and the template [1]. An interesting observation is that TM is identical to the maximal order-statistics filter (OSF)—morphological erosion—in the absence of any degradation [4]. This observation gives rise to an important

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question: Does this relationship extend to the OSF and TM in the presence of arbitrary degradation?

A mild variation of the template matching problem is provided by the composite template—a template of the interior and exterior of the object. The corresponding modification of TM yields composite template matching (CTM)—the maxima of the cross-correlation of the image and the composite template [1]. A similar modification of the OSF is used to represent the composite order-statistics filter (COSF). Once again, we observe that CTM is identical to the maximal COSF—hit-or-miss transform—in the absence of any degradation [4]. The essential question posed earlier remains: Does this relationship extend to the COSF and CTM in the presence of arbitrary degradation?

An important aspect of our investigation is focused on the optimal implementation of morphological filters in template matching. It is well known that TM provides the maximum *a posteriori* filter for the template matching problem in the presence of independent and identically distributed noise degradation [1]. In view of the extension of the OSF and TM to the composite template matching problem we may ponder: Can a weighted version of CTM provide the maximum *a posteriori* filter for the template matching problem in the presence of composite (mixed) independent and identically distributed noise degradation? Is the relationship of the weighted version of the OSF and TM—in both the standard and the composite implementations—preserved in the presence of arbitrary degradation?

In this paper, we investigate methods for optimal morphological pattern recognition. As we shall demonstrate throughout our investigation, an affirmative response is presented to all of the questions posed in our discussion above. As a consequence of our investigation, the relationship of the OSF and TM—in both the standard as well as the weighted and composite implementations—is established. This relationship is based on the thresholded cross-correlation representation of the OSF. The optimal order and weights of the OSF for pattern recognition are subsequently derived. An additional outcome of this representation is a fast method for the implementation of the OSF.

II. Weighted Order-Statistics Filters (WOSF)

In this section, we investigate the problem of optimal template matching in the presence of independent and identically distributed noise degradation.

A. Definition

Let us consider a binary image $f(x) \in \{0, 1\}$, for every $x \in \mathcal{Z}^n$. The *weight* $w(x)$ is used to denote a gray-level image $w(x) \in \mathcal{N}$, for every $x \in \mathcal{Z}^n$.

The m^{th} *weighted order-statistics filter* (WOSF) $[m^{th}WOSF(f;w)](x)$ of the binary image $f(x)$ with respect to the weight $w(x)$ is given by

$$[m^{th}WOSF(f;w)](x) = [m^{th}OS([f(x+y) \star w(y) \colon y \in \mathcal{Z}^n])] \tag{1}$$

for every $x \in \mathcal{Z}^n$, and for every $m = 1, 2, \cdots, W$, where $W = \Sigma_{x \in \mathcal{Z}^n} \; w(x)$.

The m^{th} *order-statistics filter* (OSF) $[m^{th}OSF(f;w)](x)$ of the binary image $f(x)$ with respect to the weight $w(x)$ is an important special case of the WOSF $[m^{th}WOSF(f;w)](x)$, where a binary image is used for the weight $w(x) \in \{0, 1\}$, for every $x \in \mathcal{Z}^n$.[1]

The following are some examples of the OSF:

1) *Morphological Dilation:*
$[f \oplus w](x) = [1^{st}OSF(f;w)](x)$;

2) *Median Filter:*
$[med(f;w)](x) = [(W+1/2)^{th}OSF(f;w)](x)$;

3) *Morphological Erosion:*
$[f \ominus w](x) = [W^{th}OSF(f;w)](x)$;

where W is odd.

B. Thresholded Cross-Correlation Representation

The principle motivation in our representation of the WOSF is based on its decomposition into efficient nonlinear and linear operations. This representation has the potential of tremendous improvement in the efficiency of the implementation of the WOSF. A similar notion has also been presented by several investigators for the representation of various nonlinear filters, e.g., morphological dilation [9], OSF [6], [11], [12], hit-or-miss transform [11], rank hit-or-miss transform [11], morphological skeleton [6], and sorting [12].[2]

The *threshold* $h_m(x)$ of the scalar x by the threshold m is given by

$$h_m(x) = \begin{cases} 1, & x \geq m; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Let us use $[f_1 \otimes f_2](x)$ to denote the *cross-correlation* of the images $f_1(x)$ and $f_2(x)$[7].

It can be easily shown that the WOSF $[m^{th}WOSF(f;w)](x)$ of the binary image $f(x)$ with respect to the weight $w(x)$ is given by

$$[m^{th}WOSF(f;w)](x) = h_m([f \otimes w](x)) \tag{3}$$

for every $x \in \mathcal{Z}^n$, and for every $m = 1, 2, \cdots, W$.

Let us use $\mathcal{F}[\cdot]$ to denote the Fourier transform operator [3]. The Fourier transforms $F(\omega)$ and $W(\omega)$ of the image $f(x)$ and the weight $w(x)$ are given by $F(\omega) = \mathcal{F}[f(x)]$ and $W(\omega) = \mathcal{F}[w(x)]$, respectively.

An alternative representation of the WOSF $[m^{th}WOSF(f;w)](x)$ of the binary image $f(x)$ with respect to the weight $w(x)$ is given by

$$[m^{th}WOSF(f;w)](x) = h_m(\mathcal{F}^{-1}[F(\omega) \cdot W^*(\omega)]) \tag{4}$$

for every $x \in \mathcal{Z}^n$, and for every $m = 1, 2, \cdots, W$, where $W^*(\omega)$ denotes the complex-conjugate of $W(\omega)$.

The thresholded cross-correlation representation can consequently be used to provide an efficient implementation of the WOSF. This implementation relies on efficient methods for the implementation of the Fourier transform (and its inverse) based on the FFT [3].

C. Optimal Morphological Pattern Recognition: TM

The task of optimal template matching in the presence of independent and identically distributed noise degradation is posed as a solution to a hypothesis testing problem.

Let us consider a binary image $f(x)$ corresponding to the shifted version of the binary template $t_i(x)$ corrupted by a Bernoulli point process $\mathcal{B}(p)$ that preserves the original value of the image with probability p, for $i = 0, 1$ [8].

The hypotheses $H_0(y)$ and $H_1(y)$ are given by

$$\begin{aligned} H_0(y)\colon \quad & f(x) = t_0(x-y) + n(x) \\ H_1(y)\colon \quad & f(x) = t_1(x-y) + n(x) \end{aligned} \tag{5}$$

where $n(x) \sim \mathcal{B}(p)$, for every $x \in \mathcal{Z}^n$, and for every $y \in \mathcal{Z}^n$.

[1]The binary image represented by the weight $w(x) \in \{0, 1\}$ is often referred to as the *structuring element* (or the *window*) [2], [5].

[2]Some traces of this idea can also be found in [4].

The optimality criteria used is the minimum probability of error decision rule $d(f)$—maximum *a posteriori* filter—given by

$$d(f) = \arg \max_{[i;\, y]} p(t_i(x-y)/f(x)) \tag{6}$$

where $p(t_i(x-y)/f(x))$ is used to denote the posterior conditional density function [10].

Let us assume that the hypotheses $H_0(y)$ and $H_1(y)$ have equal priors; i.e., $p(t_0(x-y)) = p(t_1(x-y))$, for every $y \in \mathcal{Z}^n$, and for $i = 0, 1$. Let us also assume (without loss of generality) that the probability p of the Bernoulli point process $\mathcal{B}(p)$ is such that $p \geq 1/2$.[3]

It can be easily shown that the solution of the hypothesis testing problem posed is equivalent to the minimum k-norm error estimate, i.e.,

$$d(f) = \arg \min_{[i;\, y]} \sum_{x \in \mathcal{Z}^n} |f(x) - t_i(x-y)|^k \tag{7}$$

for every $k > 0$.[4]

Let us now assume that the energy of the binary templates $t_i(x)$, $i = 0, 1$, is constant; i.e., $E = \Sigma_{x \in \mathcal{Z}^n} t_i^2(x)$, for $i = 0, 1$.

The classical solution to the least-squares error estimate $(k = 2)$ is provided by *template matching* (TM) $[TM(f; t_i)](y)$ given by

$$[TM(f; t_i)](y) = h_M([f \otimes t_i](y)) \tag{8}$$

where $M = \max\{[f \otimes t_i](y)\colon y \in \mathcal{Z}^n; i = 0, 1\}$, for every $y \in \mathcal{Z}^n$, and for $i = 0, 1$ [7].

It is important to note that the maxima of the cross-correlation $[f \otimes t_i](y)$ of the image $f(x)$ and the template $t_i(x)$ is attained if and only if $f(x) = \alpha t_i(x-y)$, for any scalar α, and for $i = 0, 1$ [7].

As a result of our discussion, we observe that TM $[TM(f; t_i)](y)$ provides the solution to the minimum probability of error—maximum *a posteriori*—hypothesis testing problem, in the presence of independent and identically distributed noise degradation.

D. Optimal Morphological Pattern Recognition: OSF

The introduction of the WOSF had been motivated by its potential application in the template matching problem. Our attention shall now focus on the characterization of the optimal order and weights of the WOSF for template matching.

From the discussion in the previous sections, we observe that TM $[TM(f; t_i)](y)$ is equivalent to the OSF $[\hat{m}^{th} OSF(f; \hat{w})](x)$, where $\hat{w}(x) = t_i(x)$ and $\hat{m} = \max\{[f \otimes \hat{w}](y)\colon y \in \mathcal{Z}^n; i = 0, 1\}$, for $i = 0, 1$ [see (3) and (8)].

As an immediate consequence of this relationship, we observe that the OSF $[\hat{m}^{th} OSF(f; \hat{w})](x)$ provides the solution to the minimum probability of error—maximum *a posteriori*—hypothesis testing problem, in the presence of independent and identically distributed noise degradation.

III. Weighted Composite Order-Statistics Filters (WCOSF)

In this section, we investigate the problem of optimal template matching in the presence of composite (mixed) independent and identically distributed noise degradation.

[3] This assumption imposes no restriction since its satisfaction is guaranteed by the degraded binary image $f(x)$ or its complement.

[4] The result presented is obtained by a generalization of the derivation of the equivalence of the hypothesis testing problem posed and the least-absolute error estimate $(k = 1)$ in [1].

A. Definition

Let us consider a vector $[f_k\colon k = 1, 2, \cdots, N]$ of scalars f_k, $k = 1, 2, \cdots, N$. Assume (without loss of generality) that $f_1 \leq f_2 \leq \cdots \leq f_N$. The m^{th} *order-statistics* (OS) $[m^{th} OS([f_k\colon k = 1, 2, \cdots, N])]$ of the vector $[f_k\colon k = 1, 2, \cdots, N]$ is given by $[m^{th} OS([f_k\colon k = 1, 2, \cdots, N])] = f_m$, for every $m = 1, 2, \cdots, N$.

Let us consider a binary image $f(x) \in \{-1, +1\}$, for every $x \in \mathcal{Z}^n$. The *interior weight* $w_1(x)$ and *exterior weight* $w_2(x)$ are used to denote gray-level images $w_1(x) \in \mathcal{N}$ and $w_2(x) \in \mathcal{N}$, for every $x \in \mathcal{Z}^n$. The *replication* $[f \star w_i(x)]$ of the scalar f by the weight $w_i(x)$ is given by $[f \star w_i(x)] = [f_1, f_2, \cdots, f_{w_i(x)}]$, for every $x \in \mathcal{Z}^n$, and for $i = 1, 2$.

The m^{th} *weighted composite order-statistics filter* (WCOSF) $[m^{th} WCOSF(f; w_1; w_2)](x)$ of the binary image $f(x)$ with respect to the weights $w_1(x)$ and $w_2(x)$ is given by

$$\begin{aligned}[m^{th} WCOSF(f; w_1; w_2)](x) &= [m^{th} OS([f(x+y) \star w_1(y);\\ &\quad - f(x+y) \star w_2(y)\colon y \in \mathcal{Z}^n])]\end{aligned} \tag{9}$$

for every $x \in \mathcal{Z}^n$, and for every $m = -W, \cdots, -1, 0, 1, \cdots, W$, where $W = \Sigma_{x \in \mathcal{Z}^n} w_1(x) + w_2(x)$.

The m^{th} *composite order-statistics filter* (COSF) $[m^{th} COSF(f; w_1; w_2)](x)$ of the binary image $f(x)$ with respect to the weights $w_1(x)$ and $w_2(x)$ is an important special case of the WCOSF $[m^{th} WCOSF(f; w_1; w_2)](x)$, where binary images are used for the weights $w_1(x) \in \{0, 1\}$ and $w_2(x) \in \{0, 1\}$, for every $x \in \mathcal{Z}^n$.[5] [6]

A well known example of the COSF is the *hit-or-miss transform* $[f \odot (w_1; w_2)](x)$ given by $[f \odot (w_1; w_2)](x) = [W^{th} COSF(f; w_1; w_2)](x)$, for every $x \in \mathcal{Z}^n$.[7]

B. Thresholded Weighted Cross-Correlation Representation

We shall again initiate our investigation by providing the thresholded weighted cross-correlation representation of the WCOSF.

The *threshold* $\overline{h}_m(x)$ of the scalar x by the threshold m is given by

$$\overline{h}_m(x) = \begin{cases} +1, & x \geq m; \\ -1, & \text{otherwise.} \end{cases} \tag{10}$$

The *composite weight* $w(x)$ of the weights $w_1(x)$ and $w_2(x)$ is given by $w(x) = w_1(x) - w_2(x)$, for every $x \in \mathcal{Z}^n$.

It can be easily shown that the WCOSF $[m^{th} WCOSF(f; w_1; w_2)](x)$ of the binary image $f(x)$ with respect to the weights $w_1(x)$ and $w_2(x)$ is given by

$$[m^{th} WCOSF(f; w_1; w_2)](x) = \overline{h}_m([f \otimes w](x)) \tag{11}$$

for every $x \in \mathcal{Z}^n$, and for every $m = -W, \cdots, -1, 0, 1, \cdots, W$.

The corresponding representation of the WCOSF $[m^{th} WCOSF(f; w_1; w_2)](x)$ of the binary image $f(x)$ with respect to the weights $w_1(x)$ and $w_2(x)$ is given by

$$[m^{th} WCOSF(f; w_1; w_2)](x) = \overline{h}_m(\mathcal{F}^{-1}[F(\omega) \cdot W^*(\omega)]) \tag{12}$$

for every $x \in \mathcal{Z}^n$, and for every $m = -W, \cdots, -1, 0, 1, \cdots, W$.

[5] The binary images represented by the interior and exterior weights, $w_1(x) \in \{0, 1\}$ and $w_2(x) \in \{0, 1\}$, are often referred to as the *interior* and *exterior structuring elements* (*windows*) and [2], [5].

[6] The COSF is also referred to as the *rank hit-or-miss transforms* [11].

[7] The *generic shape recognition transform* $[GSR(f; t; w)](x)$ of the binary image $f(x)$ with respect to the binary template $t(x)$ and the binary window $w(x)$, such that $t(x) \leq w(x)$, is a special case of the hit-or-miss transform, given by $[GSR(f; t; w)](x) = [f \odot (t; w - t)](x)$, for every $x \in \mathcal{Z}^n$ [4].

According to the rationale provided earlier in our presentation, we observe that the thresholded weighted cross-correlation representation can be used for the efficient implementation of the WCOSF. An extension of the conclusions presented based on the computer simulation experiments can be used to determine that the resulting improvement in the computational efficiency of the proposed implementation of the COSF rises as the cardinality of the interior and exterior structuring elements increases.[8]

C. Optimal Morphological Pattern Recognition: WCTM

A modification of the optimal template matching in the presence of composite (mixed) independent and identically distributed noise degradation is posed as a solution to a hypothesis testing problem.

Let us consider a binary image $f(x) \in \{-1, +1\}$ corresponding to the shifted version of the binary template $t_i(x) \in \{-1, +1\}$ corrupted by a composite (mixed) Bernoulli point process $\mathcal{B}(p_1, p_2)$ that preserves the original value of the image in the interior of the template with probability p_1 and preserves the original value of the image in the exterior of the template with probability p_2, for $i = 0, 1$ [8].

The hypotheses $H_0(y)$ and $H_1(y)$ are given by

$$\begin{aligned} H_0(y)\colon &\quad f(x) = t_0(x-y) + n(x) \\ H_1(y)\colon &\quad f(x) = t_1(x-y) + n(x) \end{aligned} \tag{13}$$

where $n(x) \sim \mathcal{B}(p_1, p_2)$, for every $x \in \mathcal{Z}^n$, and for every $y \in \mathcal{Z}^n$.

The optimality criteria remains the minimum probability of error decision rule $d(f)$—maximum *a posteriori* filter—given by

$$d(f) = \arg \max_{[i;\, y]} p(t_i(x-y)/f(x)) \tag{14}$$

where $p(t_i(x-y)/f(x))$ is used to denote the posterior conditional density function [10].

Let us assume that the hypotheses $H_0(y)$ and $H_1(y)$ have equal priors, i.e., $p(t_0(x-y)) = p(t_1(x-y))$, for every $y \in \mathcal{Z}^n$, and for $i = 0, 1$.

It can be easily shown that the solution of the hypothesis testing problem posed is equivalent to the weighted minimum k-norm error estimate, i.e.,

$$d(f) = \arg \min_{[i;\, y]} \sum_{x \in \mathcal{Z}^n} w(x-y)|f(x) - t_i(x-y)|^k \tag{15}$$

where $w(x) = \log(p_1/1-p_1)h_0(t_i(x)) + \log(p_2/1-p_2)h_0(-t_i(x))$, for every $x \in \mathcal{Z}^n$, and for every $k > 0$.

Let us now assume that the weighted energy of the binary templates $t_i(x), i = 0, 1$, is constant; i.e., $E_t = \Sigma_{x \in \mathcal{Z}^n}\, w(x)t_i^2(x)$, for $i = 0, 1$. We shall also assume that the weighted energy of the binary image $f(x)$ is constant; i.e., $E_f = \Sigma_{x \in \mathcal{Z}^n}\, w(x-y)f^2(x)$, for every $y \in \mathcal{Z}^n$.[9]

The solution to the weighted least-squares error estimate $(k = 2)$ is provided by *weighted composite template matching* (WCTM) $[WCTM(f; t_i; w)](y)$ given by

$$[WCTM(f; t_i; w)](y) = \overline{h}_M([f \otimes (wt_i)](y)) \tag{16}$$

and $M = \max\{[f \otimes (wt_i)](y)\colon y \in \mathcal{Z}^n; i = 0, 1\}$, for every $y \in \mathcal{Z}^n$, and for $i = 0, 1$.

We once again note that the maxima of the cross-correlation $[f \otimes (wt_i)](y)$ of the image $f(x)$ and the weighted template $w(x)t_i(x)$ is achieved if and only if $f(x) = \alpha w(x-y)t_i(x-y)$, for any scalar α, and for $i = 0, 1$.

As a consequence of this discussion, we observe that WCTM $[WCTM(f; t_i; w)](y)$ provides the solution to the minimum probability of error—maximum *a posteriori*—hypothesis testing problem, in the presence of composite (mixed) independent and identically distributed noise degradation.

D. Optimal Morphological Pattern Recognition: WCOSF

The investigation of the optimal implementation of the WCOSF for template matching shall be the focus of the remainder of our presentation. We provide a characterization of the optimal order and weights of the WCOSF for template matching.

From the discussion in the previous sections, we observe that WCTM $[WCTM(f; t_i; w)](y)$ is equivalent to the WCOSF $[\hat{m}^{th}WCOSF(f; \hat{w}_1; \hat{w}_2)](x)$, where $\hat{w}_1(x) = \log(p_1/1-p_1)h_0(t_i(x))$ and $\hat{w}_2(x) = \log(p_2/1-p_2)h_0(-t_i(x))$, and $\hat{m} = \max\{[f \otimes \hat{w}](y)\colon y \in \mathcal{Z}^n;\ i = 0, 1\}$, and $\hat{w}(x) = \hat{w}_1(x) - \hat{w}_2(x)$, for $i = 0, 1$ (see (11) and (16)).[10]

This relationship implies that the WCOSF $[\hat{m}^{th}WCOSF(f; \hat{w}_1, \hat{w}_2)](x)$ provides the solution to the minimum probability of error—maximum *a posteriori*—hypothesis testing problem, in the presence of composite (mixed) independent and identically distributed noise degradation.

IV. Summary

In this paper, we investigated methods for the implementation of morphological filters in template matching. The relationship of the OSF and TM—in both the standard as well as the weighted and composite implementations—had been established. This relationship is based on the thresholded cross-correlation representation of the OSF. As an outcome of this representation, a fast method for the implementation of the OSF is provided. The optimality of TM as the solution to the minimum probability of error—maximum *a posteriori*—hypothesis testing problem was presented. This presentation was subsequently used to provide the characterization of the optimal order and weights of the OSF in template matching. The resulting OSF had been demonstrated to be identical to TM and consequently yields the optimal solution to the minimum probability of error—maximum *a posteriori*—hypothesis testing problem. These results were presented for the implementation of the OSF and TM in template matching in the presence of independent and identically distributed noise degradation. A modification of these results has also been derived for the implementation of the WCOSF and WCTM in template matching in the presence of composite (mixed) independent and identically distributed noise degradation.

References

[1] R. O. Duda and P. E. Hart, *Pattern Classification and Scene Analysis*. New York: Wiley, 1973.

[2] J. Serra, *Image Analysis and Mathematical Morphology*. New York: Academic, 1982.

[3] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[4] T. R. Crimmins and W. M. Brown, "Image algebra and automatic shape recognition," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 21, pp. 60–69, 1985.

[5] C. R. Giardina and E. R. Dougherty, *Morphological Methods in Image and Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[6] J. E. Mazille, "Mathematical morphology and convolutions," *J. Microscopy*, vol. 156, pp. 3–13, 1988.

[7] A. K. Jain, *Fundamentals of Digital Image Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[8] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, New York: McGraw-Hill, 1991.

[8]The manifestation of this phenomenon implies that the improvement in the computational efficiency of the proposed implementation of the WCOSF rises as the region of support of the interior and exterior weights increases.

[9]In practice, this constraint can be alleviated by restricting our attention to a subset of the discrete space [1].

[10]Notice that, the optimal composite weight $\hat{w}(x)$ is identical to the weighted template $w(x)t_i(x)$; i.e., $\hat{w}(x) = w(x)t_i(x)$, for every $x \in \mathcal{Z}^n$, and for $i = 0, 1$.

[9] B. Kisačanin and D. Schonfeld, "A fast threshold linear convolution representation of morphological operations," *IEEE Trans. Image Processing*, vol. 3, pp. 455–457, July 1994.
[10] H. V. Poor, *An Introduction to Signal Detection and Estimation*, New York: Springer-Verlag, 1994.
[11] H. J. A. M. Heijmans, *Morphological Image Operators*. San Diego, CA: Academic, 1994.
[12] D. Schonfeld, "Fast parallel nonlinear filtering based on the FFT," in *Proc. Int. Conf. Digital Signal Processing*, vol. 1, Limassol, Cyprus, 1995, pp. 338–341.

Examples of Bivariate Nonseparable Compactly Supported Orthonormal Continuous Wavelets

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***Abstract*—We give many examples of bivariate nonseparable compactly supported orthonormal wavelets whose scaling functions are supported over [0, 3] × [0, 3]. The Hölder continuity properties of these wavelets are studied.**

***Index Terms*—Compact support, continuous, nonseparable, orthonormal, wavelet.**

I. INTRODUCTION

Univariate wavelets have found successful applications in signal processing since wavelet expansions are more appropriate than conventional Fourier series to represent the abrupt changes in nonstationary signals. To apply wavelet methods to digital image processing, we have to construct vibariate wavelets. The most commonly used method is the tensor product of univariate wavelets. This construction leads to a separable wavelet which has a disadvantage of giving a particular importance to the horizontal and vertical directions. Much effort (cf., e.g., [1]–[3]) has been spent on constructing nonseparable bivariate wavelets. In this paper, we construct vibariate nonseparable compactly supported orthonormal wavelets based on the commonly used uniform dilation matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Let

$$m_o(\omega) := m_0(\omega_1, \omega_2) = \sum_{0 \le j \le p, 0 \le k \le q} c_{j,k} \exp(i(j\omega_1 + k\omega_2))$$

be a trigonometric polynomial. We will construct m_0 which satisfies the following requirements: 1°: $m_0(0,0) = 1$; 2°: $\Sigma_{j=0}^3 \ |m_0(\omega + \pi_j)|^2 = 1$ with $\pi_0 = (0,0)$, $\pi_1 = (\pi, 0)$ $\pi_2 = (0, \pi)$, and $\pi_3 = (\pi, \pi)$. Let $\hat{\phi}(\omega) = \Pi_{k=1}^{\infty} \ m_0(\omega/2^k)$ be generated by m_0. Then 1° implies the convergence of this infinite product and hence $\hat{\phi}$ is a well-defined continuous function. 2° implies $\hat{\phi} \in L_2(\mathbf{R}^2)$. Thus, $\phi \in L_2(\mathbf{R}^2)$ by Plancheral's Theorem. For a fixed ordering which maps bi-integers (0, 0) ≤ (j, k) ≤ (p, q) into positive integers $\{1, 2, \cdots, N\}$ with $N = (p+1)(q+1)$, let A be a matrix of size $N \times N$ with entries

$$A_{k_1,k_2;\ell_1,\ell_2} = 4 \sum_{j_1,j_2} c_{j_1,j_2} \overline{c_{(j_1,j_2)+(k_1,k_2)-2(\ell_1,\ell_2)}}$$

for (0, 0) ≤ $(k_1, k_2, (\ell_1, \ell_2)$ ≤ (p, q). In order to make $\{\phi(x - k_1, y - k_2), (k_1, k_2) \in Z^2\}$ to be an orthonormal set, we need to have the bivariate generalizationof the Lawton condition 3° (cf. [4]). One is a nondegenerate eigenvalue of A. We then further study the coefficients of m_0 such that 4°: $\phi \in C^{\gamma}(\mathbf{R}^2)$ with $\gamma \ge 0$. After these preparations, we shall construct $m_\nu, \nu = 1, 2, 3$, such that 5°: $\sum_{j=0}^3 \ m_\mu(\omega + \pi_j)\overline{m_\nu(\omega + \pi_j)} = \delta_{\mu,\nu}$, $\mu, \nu = 0, 1, 2, 3$. To make m_0 to be a low-pass filter, we require that m_0 have a factor $(1 + e^{i\omega})$ $(1 + e^{i\omega_2})$. That is, 6°: $m_0(\pi, \omega_2) = 0 = m_0(\omega_1, \pi)$ for all $(\omega_1, \omega_2) \in [-\pi, \pi]$. For $p = q = 3$, we are able to give a complete solution set of all m_0 satisfying 1°, 2°, and 6°. We identify many families of solutions which further satisfy 3° and 4°. For example, a tensor product of Daubechies' scaling function $_2\phi$ is included. It is known that $_2\phi(x_1)_2\phi(x_2) \in C^{\alpha}(\mathbf{R}^2)$ with $\alpha \ge 0.5$ [5]. We can expect other solutions to have certain Hölder's exponents. We study the regularity of those filters. Finally, we construct m_ν to satisfy 5° for any given m_0 satisfying 1° and 2°. In Section III, we present some numerical experiments using our nonseparable wavelets.

II. CONSTRUCTION OF SCALING FUNCTIONS AND WAVELETS

Rewrite $m_0(\omega_1, \omega_2)$ as $m(x, y) = \Sigma_{0 \le j \le p, 0 \le k \le q} \ c_{j,k} x^j y_k$ with $x = e^{i\omega_1}$ and $y = e^{i\omega_2}$. Also write $m(x, y)$ in its polyphase form: $m(x,y) = f_0(x^2, y^2) + s f_1(x^2, y^2) + y f_2(x^2, y^2) + xy f_3(x^2, y^2)$. It is well-known that a polynomial m satisfying 2° is equivalent to

$$|f_0|^2 + |f_1|^2 + |f_2|^2 + |f_3|^2 = \tfrac{1}{4}. \tag{1}$$

From now on, we only consider $p = q = 3$. Thus, we write $f_\nu(x, y) = a_\nu + v_\nu x + c_\nu y + d_\nu xy$, ν = 0, 1, 2, 3. We now present one of the main results in this paper.

Theorem 2.1.: Let

$$m(x,y) = \frac{(1+x)(1+y)}{16}(a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2 + a_{02}y^2) \tag{2}$$

with

$$\begin{aligned}
a_{00} &= 1 + \sqrt{2}(\cos\alpha + \cos\beta) + 2\cos\theta\cos\xi \\
a_{10} &= \sqrt{2}(\sin\alpha - \cos\alpha) - 2\cos\theta\cos\xi + 2\cos\theta\sin\xi \\
a_{01} &= \sqrt{2}(\sin\beta - \cos\beta) - 2\cos\theta\cos\xi + 2\sin\theta\cos\eta \\
a_{11} &= 2(\cos\theta\cos\xi + \sin\theta\sin\eta - \cos\theta\sin\xi - \sin\theta\cos\eta) \\
a_{20} &= 1 + \sqrt{2}(\cos\beta - \sin\alpha) - 2\cos\theta\sin\xi \\
a_{02} &= 1 + \sqrt{2}(\cos\alpha - \sin\beta) - 2\sin\theta\cos\eta \\
a_{21} &= \sqrt{2}(\sin\beta - \cos\beta) - 2\sin\theta\sin\eta + 2\cos\theta\sin\xi \\
a_{12} &= \sqrt{2}(\sin\alpha - \cos\alpha) - 2\sin\theta\sin\eta + 2\sin\theta\cos\eta \\
a_{22} &= 1 - \sqrt{2}(\sin\alpha + \sin\beta) + 2\sin\theta\sin\eta.
\end{aligned} \tag{3}$$

Then, $m(x, y)$ satisfies 2° if $\alpha, \beta, \theta, \xi, \eta$ satisfy the following:

$$\cos\theta\cos\xi + \cos\theta\sin\xi + \sin\theta\cos\eta + \sin\theta\sin\eta = 2\sin\left(\alpha + \frac{\pi}{4}\right)\sin\left(\beta + \frac{\pi}{4}\right). \tag{4}$$

Proof: It is straightforward to verify that f_0, f_1, f_2 and f_3 satisfy (1) if and only if

$$\sum_{\nu=0}^{3} (a_\nu b_\nu + c_\nu d_\nu) = 0, \quad \sum_{\nu=0}^{3} (a_\nu c_\nu + b_\nu d_\nu) = 0,$$
$$\sum_{\nu=0}^{3} a_\nu d_\nu = 0, \quad \sum_{\nu=0}^{3} b_\nu c_\nu = 0 \tag{5}$$

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