Optimal Morphological Pattern Restoration from Noisy Binary Images

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Abstract—A theoretical analysis of morphological filters for the "optimal" restoration of noisy binary images is presented. The problem is formulated in a general form and an "optimal" solution is obtained by using fundamental tools from mathematical morphology and decision theory. We consider the set-difference distance function as a measure of comparison between images, and, by using this function, we introduce the mean difference function as a quantitative measure of the degree of geometrical and topological distortion introduced by morphological filtering. We prove that the class of alternating sequential filters is a set of parametric, smoothing morphological filters that "best" preserve the crucial structure of input images in the least mean difference sense. A theory is also presented that demonstrates some important properties of the class of alternating filters and the class of alternating sequential filters and provides a theoretical justification for their use in morphological image analysis applications. A minimax estimation procedure is also proposed that allows us to obtain the "optimal" alternating sequential filter. This filter "optimally" eliminates the rough characteristics of the degradation noise while it "optimally" preserves the crucial geometrical and topological features of the noise-free pattern under consideration.

Index Terms—Image smoothing, minimax estimation, morphological image analysis, morphological modeling, nonlinear filtering, shape analysis.

I. INTRODUCTION

An important problem in image processing and analysis applications is to develop an efficient filtering procedure that restores a binary image (pattern) from its noisy version [1]–[3]. In order to devise such a filtering procedure, we have to consider two fundamental issues: 1) the filter under consideration should be effective in eliminating the noise degradation, and 2) it should be able to restore various important aspects of the shape-size content of the noise-free image under consideration, as well as to preserve its crucial geometrical and topological structure. These are two important requirements, since many algorithms for pattern analysis, which process noisy data, critically depend on an accurate geometrical and topological image description [1], [4].

A traditional approach to solving this problem is by means of linear filtering techniques. Although this is a mathematically simple approach, which is very popular among image analysts, it usually results in a distortion of many important image characteristics. A very promising alternative solution is by means of nonlinear filtering techniques, and specifically, by employing a class of nonlinear filters known as morphological filters [1]–[3], [5]–[8].

The study of morphological filters provides a useful theoretical and computational framework for an important class of shape-oriented transformations [1], [6]. Morphological filtering was introduced by Matheron [6] with the inception of morphological openings and closings. The class of alternating filters, constructed by a single composition of a morphological opening and closing, has been shown to be a useful tool in image analysis applications. For example, Maragos and Schafer [3] have demonstrated a strong relationship between the alternating filter and the median filter. Recently, Sternberg [2] introduced a new class of morphological filters, the class of alternating sequential filters, which is constructed by an alternating series of increasing size morphological openings and closings. The alternating sequential filters have been successfully used in a variety of applications, such as remote sensing [9] and medical imaging [10] (see also [1] and [2]).

In this paper, we present a theoretical and computational study of the application of morphological filters to the problem of pattern restoration from noisy and binary images. In Section II we summarize the basic theory of morphological filtering and introduce some new results that are used in subsequent sections of the paper. In Section III we use the pattern-spectrum as a morphological tool for the mathematical description of the shape-size content of a binary image, and we study the class of smoothing morphological filters that best preserve the crucial geometrical and topological structure of images. We introduce the set-difference distance function as a measure of comparison between images. By using this function, we introduce the mean difference function as a quantitative measure of the degree of geometrical and topological distortion introduced by morphological filtering. We prove that the class of alternating sequential filters is a set of parametric, smoothing morphological filters that best preserve the crucial structure of the input image in the least mean difference sense. We also present a theory that demonstrates some important properties of the class of alternating sequential filters, and we provide a theoretical justification of its use in morphological image analysis applications. Additionally, we prove that the class of alternating filters is a set of "suboptimal" parametric, smoothing morphological filters in the least mean difference sense. In Section IV we derive the optimal parametrization of the parametric morphological filters discussed in the previous section. The degradation process is assumed to be a randomly distributed binary set, known as the germ-grain model [4], [12]. By using the set-difference distance function we quantify the restoration error introduced by the use of parametric morphological filters. We compute an upper bound on the restoration error in the case of the class of alternating sequential filters, and we derive the "optimal" alternating sequential filter, among the class of alternating sequential filters, by a minimax estimation procedure. Additionally, we compute an upper bound on the restoration error in the case of the class of alternating

1 An extended summary of our results may be found in [11].

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filters, and we derive the optimal alternating filter, among the class of alternating filters, by using the same procedure. We demonstrate the efficiency of the discussed morphological filters in reconstructing binary patterns from noisy binary images by means of simulation. Finally, in Section V we conclude our paper with a brief summary of our results.

II. MORPHOLOGICAL FILTERS

In this section we summarize the basic theory of morphological filtering and present some new results that will be used later in the paper. By convention, we let $X - B$, $X + B$, $X \ominus B$, $\gamma_B(X)$, and $\phi_B(X)$ denote the set difference, erosion, dilation, opening, and closing, respectively, of a set $X$ by a structuring element $B$. We shall also denote by $\gamma_{k}(X)$ and $\phi_{k}(X)$ the opening and closing, respectively, of the set $X$ by the structuring element $kB = B \oplus B \oplus \cdots \oplus B$ ($k$ times), where $0B = \{(0,0)\}$, by convention. We have the following important definition.

Definition 1 [1]: A set transformation $\Psi(\bullet)$ is said to be a morphological filter (MF) if

$$X_{1} \subseteq X_{2} \rightarrow \Psi(X_{1}) \subseteq \Psi(X_{2}) \quad \text{(increasing)} \quad (1)$$

and

$$\Psi(\Psi(X)) = \Psi(X) \quad \text{ (idempotent).} \quad (2)$$

The following are some useful properties of MF's. Their proofs can be found in [1].

Property 1: Given a set $X$, if $\Psi(\bullet)$ is an MF, then

$$\Psi(X) \cup \Psi(\Psi(X)) = \Psi(X); \hspace{1cm} \Psi(X) \cap \Psi(\Psi(X)) = \Psi(X)$$

and

$$\Psi(X \cup \Psi(X)) \supseteq \Psi(X); \hspace{1cm} \Psi(X \cap \Psi(X)) \subseteq \Psi(X).$$

Definition 2 [1]: An MF $\Psi(\bullet)$ is a max-filter (min-filter) if

$$\Psi(X \cup \Psi(X)) = \Psi(X); \hspace{1cm} \Psi(X \cap \Psi(X)) = \Psi(X).$$

A strong filter is an MF that is both a max-filter and a min-filter. The opening $\gamma_{k}(\bullet)$ and the closing $\phi_{k}(\bullet)$ are examples of strong filters. Examples of max-filters (min-filters) are $\gamma_{k}\phi_{k}(\bullet)$ and $\phi_{k}\gamma_{k}(\bullet)$ and $\gamma_{k}\phi_{k}(\bullet)$ and $\gamma_{k}\phi_{k}(\bullet)$ [1]. Let

$$m_{k}(\bullet) = \gamma_{k}\phi_{k}(\bullet); \hspace{1cm} n_{k}(\bullet) = \phi_{k}\gamma_{k}(\bullet) \quad (3)$$

and

$$r_{k}(\bullet) = \phi_{k}\gamma_{k}(\bullet); \hspace{1cm} s_{k}(\bullet) = \gamma_{k}\phi_{k}(\bullet). \quad (4)$$

The MF's $m_{k}(\bullet)$ and $n_{k}(\bullet)$ are collectively referred to as the alternating filters (AF's). In addition, let

$$M_{k}(\bullet) = m_{k}m_{k-1} \cdots m_{1}(\bullet); \hspace{1cm} N_{k}(\bullet) = n_{k}n_{k-1} \cdots n_{1}(\bullet). \quad (5)$$

The MF's $M_{k}(\bullet)$ and $N_{k}(\bullet)$ are collectively referred to as the alternating sequential filters (ASF's) [1].

Definition 3 [1]: The dual morphological filter $\hat{\Psi}(\bullet)$ of an MF $\Psi(\bullet)$ is given by

$$\hat{\Psi}(X) = [\Psi(X)^{c}]^{c} \quad (6)$$

where $X^{c}$ denotes the set complement of set $X$.

From (6) we obtain that $\hat{\Psi}(X) = \Psi(X)$. We also have

$$\phi_{k}(X) = \gamma_{k}(X), \hspace{1cm} n_{k}(X) = m_{k}(X), \hspace{1cm} s_{k}(X) = r_{k}(X) \hspace{1cm} \text{and} \hspace{1cm} N_{k}(X) = M_{k}(X) \quad [1].$$

Property 2: Given the MF's $\Psi_{1}(\bullet), \Psi_{2}(\bullet)$, and a set $X$, if

$$\Psi_{1}(X) \subseteq \Psi_{2}(X) \hspace{1cm} \text{then} \hspace{1cm} \Psi_{1}(X^{c}) \supseteq \Psi_{2}(X^{c}).$$

Property 3: Given a set $X$

$$\gamma_{k}(X) \subseteq s_{k}(X) \subseteq m_{k}(X) \subseteq r_{k}(X) \subseteq \phi_{k}(X) \quad (7a)$$

and

$$\gamma_{k}(X) \subseteq s_{k}(X) \subseteq n_{k}(X) \subseteq r_{k}(X) \subseteq \phi_{k}(X). \quad (7b)$$

The next definition and lemma's will be useful in a later development.

Definition 4: A set $X$ is a B-connected set if for every point $x \in X$ there exists a point $y \in Z^{2}$ such that either (i) $x \in B \oplus \{y\} \subseteq X$, or (2) $x \in B \oplus \{y\} \subseteq \gamma_{B}(X)$.

Lemma 1: Given a set $X$, if $\phi_{k}(X)$ and $\phi_{k}(X)$ are $kB$-connected sets, then

$$\gamma_{k}(X) \subseteq s_{k}(X) = n_{k}(X) \subseteq m_{k}(X) \subseteq r_{k}(X) \subseteq \phi_{k}(X). \quad (8)$$

It is important to notice that, although the class of increasing transformations is closed under composition, the class of idempotent transformations is not; therefore, in general, the class of MF's is not closed under composition. Nevertheless, as will be shown in the following, the class of filters generated by compositions from a collection of MF's that form a size distribution is closed under composition. Let us consider a set $F = \{\Gamma_{i}(\bullet), \Phi_{i}(\bullet), i = 0, 1, \cdots, \lambda, \lambda \in Z^{+}\}$ of morphological transformations $\Gamma_{i}(\bullet)$ and $\Phi_{i}(\bullet)$ that forms a size distribution [1], i.e.,

$$\Gamma_{i}(\bullet), \Phi_{i}(\bullet) \text{ are increasing} \quad (9a)$$

$$\Gamma_{i}(X) \subseteq X \quad \text{ (anti-extensive); } \quad \Phi_{i}(X) \supseteq X \quad \text{ (extensive) } \quad (9b)$$

and

$$\Gamma_{i} \Gamma_{j}(\bullet) = \Gamma_{\max(i,j)}(\bullet); \hspace{1cm} \Phi_{i} \Phi_{j}(\bullet) = \Phi_{\max(i,j)}(\bullet) \quad (sieving). \quad (9c)$$

We have the following proposition.

Proposition 1: Let $F$ be the collection of transformations formed by all possible compositions generated from the collection $F$ of morphological transformations that form a size distribution. If $\Psi(\bullet) \in F$, then $\Psi(\bullet)$ is an MF.

III. PARAMETRIC SMOOTHING MORPHOLOGICAL FILTERS

The main objective of our study is the classical problem of restoration of binary images. Given a binary image $X$, let us assume that the transformation $\Phi(\bullet)$ produces a degraded binary image $Y$, i.e.,

$$Y = \Phi(X). \quad (10)$$

The problem is to derive an MF $\Psi(\bullet)$ such that the binary image $\Psi(Y)$ is the "optimal" restoration of the original binary image $X$.

4The proof of this and other lemmas is given in Appendix A.

5In [1] the collection $\{\Gamma_{i}(\bullet)\}$ is referred to as the size distribution, whereas the collection $\{\Phi_{i}(\bullet)\}$ is referred to as the anti-size distribution.

6The proof of this and other propositions is given in Appendix B.
Although this problem has been the subject of extensive research in mathematical morphology, an optimal restoration criterion is not known. The first step toward the solution to this problem is to notice that it is important that the filtering of the image \( Y \), given by (10), by the MF \( \Psi(\bullet) \) preserves some important shape-size characteristics (e.g., smooth boundaries) of the original binary image \( X \). We shall discuss this problem first.

### A. Pattern Spectrum

In order to provide a precise mathematical description of the important shape-size characteristics of an image \( X \) under consideration, we utilize the pattern spectrum.

**Definition 5** [15]: The pattern spectrum \( PS_k(X) \) of a set \( X \) in terms of the structuring element \( B \) is given by

\[
PS_k(X) = \begin{cases} 
\text{Card} \left[ \gamma_k(X) - \gamma_{k+1}(X) \right], & \text{for } k \geq 0 \\
\text{Card} \left[ \phi_{-k}(X) - \phi_{-k-1}(X) \right], & \text{for } k \leq -1
\end{cases}
\]

(11)

where \( k \in Z \) and \( \text{Card} [X] \) denotes the cardinality of a set \( X \). The pattern spectrum is a very useful tool for the shape-size description of images. The frequency spectrum, in linear systems theory, provides a measure of similarity between a function and the collection of all possible sinusoids, whereas the pattern spectrum, in mathematical morphology, provides a measure of similarity between a set and the collection of all possible structuring elements \( B \). Objects with rough boundaries contribute to the lower size part of the pattern spectrum (i.e., small values of \( k \)), whereas objects with smooth boundaries contribute to the higher size part of the pattern spectrum (i.e., large values of \( k \)). A smooth image, with respect to the structuring element \( B \), is defined to be an image \( X \), for which

\[
PS_k(X) = 0, \quad \text{for } k = -\lambda, -2, -1, 0, 1, 2, \ldots, \lambda - 1
\]

(12)

for some \( \lambda \in Z^+ \).

Let us now recall the standard linear restoration problem: a noise-free image is corrupted by additive noise and a filter is designed in order to restore the noise-free image from the noisy image by removing the undesirable effects of the additive noise. The restoration filter is usually a low-pass filter that preserves the low-pass frequency characteristics of the noise-free image, and it suppresses the (usually) high-pass frequency characteristics of the additive noise. An MF for image restoration should also preserve the important smooth characteristics of the noise-free image and suppress the (usually) rough characteristics of the contaminating noise. Selecting the appropriate MF to restore a degraded image, however, poses numerous problems due to the nonlinearity of the operations involved. In this section, we shall derive a collection of MF's that suppress the rough characteristics of the contaminating noise, and we shall study some of their important properties.

### B. Least Mean Difference Morphological Filters

From the previous discussion, it is clear that the design of the appropriate MF \( \Psi(\bullet) \) for image restoration requires the solution of two distinct, but related, problems: 1) the filtering of any image \( Y \) by the MF \( \Psi(\bullet) \) should result in a smooth image, and, 2) the output \( \Psi(Y) \) of the MF \( \Psi(\bullet) \) to any noisy image \( Y \) should be as "close" as possible to the noise-free image \( X \).

The solution to the first problem can be accomplished by considering MF's \( \Psi_s(\bullet) \) such that (see also (12)) for some \( \lambda \in Z^+ \)

\[
PS_k(\Psi_s(Y)) = 0, \quad \text{for } k = -\lambda, \ldots, -2, -1, 0, 1, 2, \ldots, \lambda - 1
\]

(13)

for every image \( Y \). We have the following lemma.

**Lemma 2**: Given a set \( X \), (12) is satisfied if and only if

\[
\gamma_k(X) = X, \quad \text{for } k = 0, 1, \ldots, \lambda
\]

(14a)

and

\[
\phi_k(X) = X, \quad \text{for } k = 0, 1, \ldots, \lambda
\]

(14b)

From Lemma 2 it is clear that, in order to obtain an MF \( \Psi_s(\bullet) \) such that the constraints of (13) are satisfied, we have to solve the system of equations

\[
\gamma_k \Psi_s(Y) = \Psi_s(Y), \quad \text{for } k = 0, 1, \ldots, \lambda
\]

(15a)

and

\[
\phi_k \Psi_s(Y) = \Psi_s(Y), \quad \text{for } k = 0, 1, \ldots, \lambda
\]

(15b)

for every image \( Y \), in terms of \( \Psi_s(\bullet) \). Observe that, if

\[
\gamma_k \Psi_s(Y) = \Psi_s(Y)
\]

(16a)

and

\[
\phi_k \Psi_s(Y) = \Psi_s(Y)
\]

(16b)

for every image \( Y \), then the system of equations (15) is satisfied exactly (this results from the sieving property of opening and closing, i.e., \( \gamma_i \gamma_j(X) = \gamma_{i+j}(X) \) and \( \phi_i \phi_j(X) = \phi_{i+j}(X) \), for every \( i, j \geq 0 \)). Unfortunately, an MF \( \Psi_s(\bullet) \) that satisfies (16) cannot be found in general, and, therefore, the constraints of (15) cannot be satisfied exactly.

A great number of constraints may be formed by various compositions of the constraints in (15), e.g., \( m_{k} \Psi_s(Y) = \Psi_s(Y) \) for \( k = 0, 1, \ldots, \lambda \). In the following, we shall denote by \( C_l \) the collection of constraints obtained by all possible compositions of the constraints in (15). Clearly, any transformation \( \Psi_s(\bullet) \) satisfying (15) must also satisfy all the constraints in \( C_l \).

The solution to the second problem is more technical. For a particular choice of the noisy image \( Y \), some of the MF's, \( \Psi_s(\bullet) \) will behave "better" (e.g., they will satisfy (13), or they will result in images "closer" to the noise-free image \( X \)) than any other MF. Since the choice of "good" MF's is a function of the image \( Y \), the solution to problem 2) is more difficult than expected. A reasonable approach to this problem is to derive the MF \( \Psi(\bullet) \) whose behavior is "uniform" among all MF's of interest for every image \( Y \).

The simultaneous solution to the previous two problems is a very difficult task, since the two individual solutions are based on conflicting requirements; therefore, a compromise is necessary at this point. One such compromise is to trade off the quality of the smoothing introduced by an MF to the quality in restoration.\(^7\) In the rest of this paper, we shall relax condition (13) and we shall consider the class of smoothing morphological filters (SMF's) \( \Psi_s(\bullet) \) that satisfy at least one constraint in \( C_l \). Since an SMF \( \Psi_s(\bullet) \) is a function of parameter \( \lambda \), it will also be referred to as the parametric morphological filter (PMF) \( \Psi_s(\bullet) \).

\(^7\)This is similar to the technique of designing finite impulse response filters via windowing, in linear digital filter theory [16]. In this case there is a tradeoff between the steeping of the transition band and the size of oscillations introduced by the filtering.
Definition 6: Given an operator \( L(\bullet) \), we say that the constraint \( L(\Psi(\bullet)) = \emptyset \) generates the transformation \( \Pi(\bullet) \) if \( L(\Pi(\bullet)) = \emptyset \).

Lemma 3: A constraint \( L(\Psi(\bullet)) = \emptyset \) in \( C_1 \) generates an MF. According to Lemma 3, each of the constraints in \( C_1 \) generates an MF. We shall denote the collection of all these MF’s by \( F_1 \). Observe that each of the MF’s in \( F_1 \) is an SMF and vice versa.

Let us now define a functional to be used for the quantitative comparison of images. We refer to this functional as the set-difference distance function.

Definition 7: Given sets \( X_1 \) and \( X_2 \), the set-difference distance function (SDDF) \( d(X_1, X_2) \) is given by

\[
d(X_1, X_2) = \text{Card}[(X_1 \cup X_2) - (X_1 \cap X_2)].
\]

Notice that the SDDF is equivalent to the Hamming distance when the sets \( X_1 \) and \( X_2 \) are represented as binary vectors [17].

Proposition 2: The SDDF is a metric, i.e., it satisfies the following properties:

i) \( d(X_1, X_2) \geq 0 \); \( d(X_1, X_2) = 0 \) if and only if \( X_1 = X_2 \) \( (18a) \)

ii) \( d(X_1, X_2) = d(X_2, X_1) \) \( (18b) \)

iii) \( d(X_1, X_2) \leq d(X_1, X_3) + d(X_2, X_3) \) \( (18c) \)

In the following lemma we shall discuss some important properties of the SDDF. Note that other metrics, defined over sets, violate this lemma, e.g., the Hausdorff metric [4], [6].

Lemma 4: Consider sets \( X_1 \), \( X_2 \), and \( Y \).

i) If \( X_1 \subseteq Y \subseteq X_2 \), then

\[
d(X_1, X_2) = d(X_1, Y) + d(Y, X_2).
\]

ii) If \( Y \subseteq X_1 \subseteq X_2 \) or \( X_1 \subseteq Y \subseteq X_2 \), then

\[
d(X_1, X_2) < d(X_1, Y) + d(Y, X_2).
\]

In order to obtain a "uniform" filter behavior over all SMF's in \( F_1 \), and for every input image \( X \), it is reasonable to consider the PMF \( \Psi(\bullet) \) in \( F_1 \) to be the least mean difference morphological filter (LMDMF) over the collection of SMF's \( \Pi(\bullet) \) in \( F_1 \), i.e., we determine the PMF \( \Psi(\bullet) \) as

\[
\Psi_M(\bullet) = \arg \min_{\Psi(\bullet) \in \Pi(\bullet)} \left\{ d(\Psi(Y), \Pi(Y)) \right\}_{Y \in \Pi(\bullet)}
\]

for every image \( Y \) where

\[
\left( d(\Psi(Y), \Pi(Y)) \right)_{\Pi(\bullet)} = \frac{1}{\text{Card}(\Pi(\bullet))} \sum_{\Pi(\bullet) \in \Pi(\bullet)} d(\Psi(Y), \Pi(Y)).
\]

Although (21) gives an exact mathematical representation to the solution of our problem, this equation cannot be solved in general. Nevertheless, the optimization problem in (21) may be solved by considering a restricted collection of images \( Y \) and a restricted collection of SMF’s \( \Pi(\bullet) \). In the following, we shall assume that all images \( Y \) under consideration are characterized by the property that \( \phi_0(Y) \) and \( \phi_1(Y) \) are \( \lambda \)-connected sets,

for \( k = 1, 2, \ldots, \lambda \). This assumption allows us to proceed with our analysis at the expense of limiting the class of images \( Y \) under consideration. We shall also derive a PMF \( \Psi_\lambda(\bullet) \) in \( F_1 \) as the LMDMF over the collection of SMF's \( \Pi(\bullet) \) in \( F_1 \), with \( F_1 \subseteq F_2 \). Finally, we shall modify our solution and obtain a PMF \( \Psi_\lambda(\bullet) \) in \( F_1 \) as the LMDMF over the collection of SMF's \( \Pi(\bullet) \) in \( F_1 \), with \( F_1 \subseteq F_2 \subseteq F_3 \). This approach allows us to study some important properties of two interesting classes of SMF's, the class of AF's and the class of ASF's.

1) Least Mean Difference Morphological Filters over \( F_2 \):

Consider a PMF \( \Psi_\lambda(\bullet) \) that satisfies the constraints (16a) and (16b), for every image \( Y \), and let \( C_\lambda \) denote the collection of the constraints obtained by all possible compositions of the constraints in (16a) and (16b). Note that \( C_\lambda \subseteq C_1 \). Let \( F_2 \) denote the collection of SMF's generated by the constraints in \( C_\lambda \).

Lemma 5: \( F_2 = \{ \gamma_\lambda(\bullet), \phi_0(\bullet), m_\lambda(\bullet), n_\lambda(\bullet), r_\lambda(\bullet), s_\lambda(\bullet) \} \).

The following definition uses Definition 2 to classify the filters in \( F_2 \) according to their geometrical properties.

Definition 8: The sets

\[
F_\lambda, \max = \{ \phi_\lambda(\bullet), m_\lambda(\bullet), r_\lambda(\bullet) \}
\]

and

\[
F_\lambda, \min = \{ \gamma_\lambda(\bullet), n_\lambda(\bullet), s_\lambda(\bullet) \}
\]

will be called the max-basis and min-basis, respectively.

Note that \( F_2 = F_\lambda, \max \cup F_\lambda, \min \). The sets \( F_\lambda, \max \) and \( F_\lambda, \min \), given by (22), will be used as the basis for the set \( F_3 \) in the following subsection. Observe that the SMF's \( \gamma_\lambda(\bullet), m_\lambda(\bullet), \) and \( s_\lambda(\bullet) \) satisfy the constraint in (16a) for every image \( Y \) (in which case \( P_{\gamma}(\Psi_\lambda(Y)) = 0 \), for \( k = 0, 1, \ldots, \lambda - 1 \) for every image \( Y \), whereas the SMF's \( \phi_0(\bullet), n_\lambda(\bullet), \) and \( r_\lambda(\bullet) \) satisfy the constraint in (16b) for every image \( Y \) (in which case \( P_{r}(\Psi_\lambda(Y)) = 0 \) for \( k = 1, 2, \ldots, \lambda \) for every image \( Y \)). When the image \( Y \) is such that \( \phi_0(Y) \) and \( \phi_1(Y) \) are \( \lambda \)-connected sets (which is our case), then the SMF's \( m_\lambda(\bullet), n_\lambda(\bullet), s_\lambda(\bullet) \), and \( r_\lambda(\bullet) \) satisfy both constraints in (16) (see also (2)–(4) and (8)). In this case, (13) is also satisfied. Finally observe that, when \( \phi_0(Y) \) and \( \phi_1(Y) \) are \( \lambda \)-connected sets, then \( n_\lambda(Y) = n_0(Y) \) and \( m_\lambda(Y) = r_\lambda(Y) \) (see Lemma 1).

We shall now derive some interesting properties for the SMF's in \( F_3 \).

Lemma 6: Consider a MF \( \Psi(\bullet) \) and its dual \( \hat{\Psi}(\bullet) \). Then

i) \( \Psi(\bullet) \in F_\lambda, \max \) if and only if \( \hat{\Psi}(\bullet) \in F_\lambda, \min \).

ii) a) \( \phi_0(\bullet) \subseteq \hat{\Psi}(\bullet) \), for all \( \Psi(\bullet) \in F_\lambda, \max \).

b) \( \phi_1(\bullet) \subseteq \hat{\Psi}(\bullet) \), for all \( \Psi(\bullet) \in F_\lambda, \min \).

iii) Consider a set \( Y \) such that \( \phi_0(Y) \) and \( \phi_1(Y) \) are \( \lambda \)-connected sets. If \( \Psi(\bullet) \in F_\lambda, \max \), then \( \hat{\Psi}(\bullet) \subseteq n_\lambda(Y) \subseteq m_\lambda(Y) \subseteq \Psi(\bullet) \).

In the next theorem we derive the SMF \( \Psi_\lambda(\bullet) \) that satisfies condition (21) over the collection of SMF's \( \Pi(\bullet) \) in \( F_3 \).

Theorem 1: Consider a set \( Y \) such that \( \phi_0(Y) \) and \( \phi_1(Y) \) are \( \lambda \)-connected sets.

i) If

\[
n_\lambda(\bullet) \subseteq \Psi_\lambda(\bullet) \subseteq m_\lambda(\bullet)
\]

then

\[
\Psi_\lambda(Y) = \arg \min_{\Psi(\bullet) \in F_\lambda, \lambda} \left( d(\Psi(Y), \Pi(Y)) \right)_{\Pi(\bullet)}
\]

8Our main conclusions are verified experimentally by considering images \( Y \) that do not satisfy the \( \lambda \)-connected assumption, therefore, demonstrating the validity of our results in practical applications.
(b) If $\Psi_3(\bullet) \in F_{3,\text{max}}$, then

$$\Psi_3(Y) = \arg \min_{\Psi(\bullet) \in F_{3,\text{max}}} (d(\Psi(Y), \Pi(Y)))_{F_3}$$

(24)

if and only if $\Psi_3(\bullet) = m_{3}\text{max}(\bullet)$.

c) If $\Psi_3(\bullet) \in F_{3,\text{min}}$, then

$$\Psi_3(Y) = \arg \min_{\Psi(\bullet) \in F_{3,\text{min}}} (d(\Psi(Y), \Pi(Y)))_{F_3}$$

(25)

if and only if $\Psi_3(\bullet) = N_{3}\text{min}(\bullet)$.

Proof: The proof is a direct consequence of Lemma 6, and it is a special case of the proof of Theorem 2.

According to Theorem 1, when $\phi_k(Y)$ and $\phi_k(Y')$ are $\lambda B$-connected sets, the AF's $m_{3}\text{max}(\bullet)$ and $n_{3}\text{max}(\bullet)$ are the LMDMF's over the collection of SMF's in $F_3$. Any SMF $\Psi_3(\bullet)$ in $F_3$ that satisfies (23) is also, in the previous sense, an optimal PMF. In particular, it is easy to prove that, when $\phi_k(Y)$ and $\phi_k(Y')$ are $\lambda B$-connected sets, for $1 \leq k \leq \lambda$, the SMF $M_{3}\text{max}(\bullet)$ and $N_{3}\text{max}(\bullet)$, given by (5), satisfy (23) and since we can also prove that $M_{3}\text{max}(\bullet), N_{3}\text{max}(\bullet) \in F_3$, these filters are the LMDMF's over the collection of SMF's in $F_3$. Finally, (24) and (25) reveal two interesting properties of the AF's in terms of the SDDF.

4) Least Mean Difference Morphological Filters over $F_3$: In this section we shall derive a PMF $\Psi_3(\bullet)$ which is the LMDMF over a larger collection of SMF's contained in $F_3$.

Let us use the notation $\Psi \rightarrow \Psi$ to denote that the filter $\Psi(\bullet)$ is composed of at least a single filter $\Psi(\bullet)$, i.e., $\Psi(X) = \Pi_1\Pi_2\Pi_3(X)$, for some filters $\Pi_1$ and $\Pi_3$.

Definition 9: The set $F_{3,\text{max}}$ contains the MF's $\Psi(\bullet)$ such that

i) Every $\Psi(\bullet) \in F_{3,\text{max}}$ is a composition of MF's in $\bigcup_{k=1}^{\lambda} F_{3,\text{max}}$.

ii) For every $\Psi(\bullet) \in F_{3,\text{max}}$, we have $\phi_k \rightarrow \Psi$.

iii) For every $\Psi(\bullet) \in F_{3,\text{max}}$, we have $m_k \rightarrow \Psi$.

The set $F_{3,\text{min}}$ contains the MF's $\Psi(\bullet)$ such that

i) Every $\Psi(\bullet) \in F_{3,\text{min}}$ is a composition of MF's in $\bigcup_{k=1}^{\lambda} F_{3,\text{min}}$.

ii) For every $\Psi(\bullet) \in F_{3,\text{min}}$, we have $\gamma_k \rightarrow \Psi$.

iii) For every $\Psi(\bullet) \in F_{3,\text{min}}$, we have $n_k \rightarrow \Psi$.

Let $F_3 = F_{3,\text{max}} \cup F_{3,\text{min}}$. Every MF in $F_3$ is generated by a constraint in $C_3$ and $F_3 \subseteq F_{3,\text{max}} \subseteq F_3$. We shall denote the set of constraints that generates the MF's in $F_{3,\text{max}}$ by $C_{3,\text{max}}$ with $C_{3,\text{max}} \subseteq C_3 \subseteq C_{3,\text{max}}$. Obtain that obtaining an exact solution to the constraints in $C_{3,\text{max}}$ is equivalent to simultaneously satisfying all of the constraints in $C_3$. Finally, observe that the SMF $M_{3}\text{max}(\bullet)$ satisfies the constraint in (16a) for every image $Y$ (in which case $PS_3(\Psi_3(Y)) = 0$, for $k = 0, 1, \ldots, \lambda - 1$, for every image $Y$), whereas the SMF $N_{3}\text{max}(\bullet)$ satisfies the constraint in (16b) for every image $Y$ (in which case $PS_3(\Psi_3(Y)) = 0$, for $k = -1, -2, \ldots, -\lambda$, for every image $Y$). The image $Y$ is such that $\phi_k(Y)$ and $\phi_k(Y')$ are $\lambda B$-connected sets, then the SMF's $M_3(\bullet)$ and $N_3(\bullet)$ satisfy both constraints in (16). In this case (13) is also satisfied.

We shall now derive some interesting properties for the MF's in $F_3$.

Lemma 7: Consider an MF $\Psi(\bullet)$ and its dual $\Psi(\bullet)$. Then

i) $\Psi(\bullet) \in F_{3,\text{max}}$ if and only if $\Psi(\bullet) \in F_{3,\text{min}}$.

ii) $M_3(\bullet) \subseteq \Psi(\bullet)$, for all $\Psi(\bullet) \in F_{3,\text{max}}$

iii) Consider a set $Y$ such that $\phi_k(Y)$ and $\phi_k(Y')$ are $kB$-connected sets for $k = 1, 2, \ldots, \lambda$.

If $\Psi(\bullet) \in F_{3,\text{max}}$, then $\Psi(Y) \subseteq N_3(Y) \subseteq M_3(Y) \subseteq \Psi(Y)$.

In the following theorems, we derive the SMF $\Psi_3(\bullet)$ which satisfies the condition in (21) over the collection of SMF's $\Pi(\bullet)$ in $F_3$.

Theorem 2: Consider a set $Y$ such that $\phi_k(Y)$ and $\phi_k(Y')$ are $kB$-connected sets for $k = 1, 2, \ldots, \lambda$.

a) If $N_3(\bullet) \subseteq \Psi_3(\bullet) \subseteq M_3(\bullet)$

then $\Psi_3(Y) = \arg \min_{\Pi(\bullet) \in F_3} (d(\Psi(Y), \Pi(Y)))_{F_3}$

(27a)

b) If $\Psi_3(\bullet) \in F_{3,\text{max}}$, then

$\Psi_3(Y) = \arg \min_{\Pi(\bullet) \in F_{3,\text{max}}} (d(\Psi(Y), \Pi(Y)))_{F_3}$

(27b)

c) If $\Psi_3(\bullet) \in F_{3,\text{min}}$, then

$\Psi_3(Y) = \arg \min_{\Pi(\bullet) \in F_{3,\text{min}}} (d(\Psi(Y), \Pi(Y)))_{F_3}$

(27c)

if and only if $\Psi_3(\bullet) = M_3(\bullet)$.

Proof: a) Observe from (21b) that

$$\begin{align*}
\langle d(\Psi(Y), \Pi(Y)), F_3 \rangle & = \frac{1}{\text{Card}[F_3]} \sum_{\Pi(\bullet) \in F_3} d(\Psi(Y), \Pi(Y)) \\
& = \frac{1}{\text{Card}[F_3]} \sum_{\Pi(\bullet) \in F_3} \left[ d(\hat{\Pi}(Y), \Psi(Y')) + d(\Psi(Y), \Pi(Y)) \right]
\end{align*}$$

(30)

where we have used Part i) of Lemma 7, (18b), and the fact that $F_3 = F_{3,\text{max}} \cup F_{3,\text{min}}$. From (30) we have

$$\langle d(\Psi(Y), \Pi(Y)), F_3 \rangle \geq \frac{1}{\text{Card}[F_{3,\text{max}}]} \times \sum_{\Pi(\bullet) \in F_{3,\text{max}}} d(\hat{\Pi}(Y), \Pi(Y)) = \min_{\Phi(\bullet) \in F_{3,\text{max}}} \langle d(\Phi(Y), \Pi(Y)), F_3 \rangle$$

(31)

since $d(\hat{\Pi}(Y), \Psi(Y')) \leq d(\hat{\Pi}(Y), \Psi(Y')) + d(\Psi(Y), \Pi(Y))$ (see also (18c)).

Consider now an MF $\Psi_3(\bullet)$ such that (27a) is satisfied. From (26) we obtain

$$\hat{\Pi}(Y) \subseteq N_3(Y) \subseteq \Psi_3(Y) \subseteq M_3(Y) \subseteq \Pi(Y)$$

for any $\Pi(\bullet) \in F_{3,\text{max}}$, which, together with (19), gives

$$d(\hat{\Pi}(Y), \Pi(Y)) = d(\hat{\Pi}(Y), \Psi_3(Y)) + d(\Psi_3(Y), \Pi(Y))$$

(32)
for any $\Pi(*) \in F_{\lambda,\max}^\star$. From (30), (31), and (32), observe that

$$(d(\Psi_\lambda(Y), \Pi(Y)))_{F_{\lambda}^\star} = \min_{\Psi_\lambda \in F_{\lambda,\max}^\star} (d(\Psi_\lambda(Y), \Pi(Y)))_{F_{\lambda}^\star}$$

(33)

which proves (27b).

b) From the results of Part a) and (33) (with $\Psi_\lambda(Y) \rightarrow M_\lambda(Y)$), we have

$$(d(M_\lambda(Y), \Pi(Y)))_{F_{\lambda}^\star} = \min_{\Psi \in F_{\lambda,\max}^\star} (d(\Psi(Y), \Pi(Y)))_{F_{\lambda}^\star}$$

$$= \min_{\Psi \in F_{\lambda,\max}^\star} (d(\Psi(Y), \Pi(Y)))_{F_{\lambda}^\star}$$

(34)

since $M_\lambda(*) \in F_{\lambda,\max}$. Let us now assume that $\Psi_\lambda(Y) \neq M_\lambda(Y)$. From (26) we have that $N_\lambda(Y) \subseteq M_\lambda(Y) \subseteq \Psi_\lambda(Y)$, which, together with (20), gives

$$d(N_\lambda(Y), M_\lambda(Y)) < d(N_\lambda(Y), \Psi_\lambda(Y)) + d(\Psi_\lambda(Y), M_\lambda(Y)).$$

(35)

From (30), (31), (34), and (35), we obtain

$$d(\Psi_\lambda(Y), \Pi(Y)))_{F_{\lambda}^\star}$$

$$= \frac{1}{2\text{Card}[F_{\lambda,\max}^\star]} \left[ d(N_\lambda(Y), \Psi_\lambda(Y)) + d(\Psi_\lambda(Y), M_\lambda(Y)) \right.$$}

$$+ \sum_{\Pi(*) \in F_{\lambda,\max}^\star \setminus \{M_\lambda(*)\}} d(\Pi(Y), \Psi_\lambda(Y)) + d(\Psi_\lambda(Y), \Pi(Y)) \bigg)$$

$$> \frac{1}{2\text{Card}[F_{\lambda,\max}^\star]} \left[ d(N_\lambda(Y), M_\lambda(Y)) 

+ \sum_{\Pi(*) \in F_{\lambda,\max}^\star \setminus \{M_\lambda(*)\}} d(\Pi(Y), \Pi(Y)) \right]$$

$$= \frac{1}{2\text{Card}[F_{\lambda,\max}^\star]} \left[ \sum_{\Pi(*) \in F_{\lambda,\max}^\star} d(\Pi(Y), \Pi(Y)) \right.$$}

$$= \min_{\Psi \in F_{\lambda,\max}^\star} (d(\Psi(Y), \Pi(Y)))_{F_{\lambda}^\star}$$

(36)

for every $\Psi_\lambda(*) \in F_{\lambda,\max}^\star$ such that $\Psi_\lambda(*) \neq M_\lambda(*)$. To prove (36) we also used the fact that $d(\Pi(Y), \Pi(Y)) \leq d(\Pi(Y), \Psi(Y)) + d(\Psi(Y), \Pi(Y))$ (see also (18c)). Equations (34) and (36) complete the proof.

c) By duality, the proof is similar to the proof of Part b). □

According to Theorem 2, when $\phi_\lambda(Y)$ and $\phi_\lambda(Y')$ are $kB$-connected sets, for $1 \leq k \leq \lambda$, the ASF's $N_i(*)$ and $N_i(*)$ are the LMDMF's over the collection of SMF's in $F_{\lambda}^\star$. Any PMF $\Psi_\lambda(*)$ in $F_{\lambda}$ that satisfies (27) is also an optimal PMF, in the previous sense. Note that the AF's $n_{\lambda}(*)$ and $n_{\lambda}(*)$ violate (27) and, therefore, are not the LMDMF's over the collection of SMF's in $F_{\lambda}^\star$. Finally, (28) and (29) reveal two interesting properties of the ASF's in terms of the SDDF.

In Fig. 1 we demonstrate the smoothing behavior of some PMF's of interest, in terms of the pattern spectrum of their output. Fig. 1(a) depicts a $256 \times 256$ binary image, whereas three filtered versions of this image are depicted in Figs. 1(b)–(d). The corresponding pattern spectra are depicted in Figs. 1(c)–(h).

Observe that all filters have a "cutoff" frequency at $\lambda = 5$, and that filter $N_3(*)$ behaves "better" than the filters $n_3(*)$ and $r_3(*)$.

IV. OPTIMAL SMOOTHING MORPHOLOGICAL FILTERS

In the previous section, we discussed the problem of determining MF's that "best" preserve the smoothness of binary patterns. We derived a class of PMF's $\{\Psi_\lambda(*)\}$ that satisfies this property in the mean difference sense. Our objective in this section is to select the "best" value $\lambda$, of parameter $\lambda$, which results in an "optimal" MF $\Psi_\lambda(*)$ for image restoration. A similar problem has been addressed in [18] for the removal of speckle noise from radar images. We shall mainly consider the "optimal" parametrization of the class of ASF's. At times, however, it may be desirable to reduce the computational complexity of the restoration process. For this reason, we shall also consider the "optimal" parametrization of the class of AF's.

A. Degradation Process

The degradation transformation $\Phi(*)$, considered in the previous section, has been completely arbitrary. In this section, we shall restrict our attention to a particular model for $\Phi(*)$.

The output $Y$ of the degradation transformation $\Phi(*)$ will be considered to be given by

$$Y = (X - N_i) \cup N_2$$

(37)

where

$$N_i = \bigcup_{n=1,2,\ldots} C_{i,n} \oplus \{x_{i,n}\}, \quad i = 1, 2.$$  

(38)

The model given by (38) is known as the germ-grain model [4, 12]. In this case $\{C_{i,n}, n = 1, 2, \ldots\}$ is a sequence of sets, known as the primary grains, whereas $\{x_{i,n}, n = 1, 2, \ldots\}$ is a sequence of sites, known as the germs, which are randomly distributed in $Z^2$, e.g., a Bernoulli point process.\(^\text{10}\) Observe that the sequence of germs $\{x_{i,n}, n = 1, 2, \ldots\}$ indicates the "centers" of the primary grains $\{C_{i,n}, n = 1, 2, \ldots\}$ in the noise

\(^\text{10}\)Occasionally the sequence of germs $\{x_{i,n}, n = 1, 2, \ldots\}$ is distributed according to a Poisson point process. In this case, the germ-grain model is known as the Boolean model [4, 12].
process $N_i$, $i = 1, 2$. Let $H$ denote the SQUARE structuring element depicted in Fig. 2(a). The following definition and property will be useful.

**Definition 10:** The sets $X_1$ and $X_2$ are said to be disconnected sets if $X_1 \oplus H \cap X_2 = \emptyset$ (or, equivalently, $X_1 \cap X_2 \oplus H = \emptyset$).

**Property 4:** If $X_1$ and $X_2$ are disconnected sets, then

$$
\gamma_k(X_1 \cup X_2) = \gamma_k(X_1) \cup \gamma_k(X_2).
$$

(39)

It is a common practice to assume that $\{C_{i,n} \oplus \{x_{i,n}\}, \ n = 1, 2, \cdots\}$ is a sequence of nonoverlapping primary grains [12]. In this paper, we shall assume that $\{C_{i,n} \oplus \{x_{i,n}\}, \ n = 1, 2, \cdots\}$ is a sequence of disconnected sets, in the sense of Definition 10. This assumption is more appropriate in the discrete space case.

The next definition will also be useful.

**Definition 11:** Given a metric $\bar{d} : Z^2 \times Z^2 \rightarrow \mathbb{R}^+$, a radial set is a set $B_r$ such that $\{(0,0)\} \in B_r$, and

$$
2kB_r \oplus \{x\} = \{y : \bar{d}(x, y) \leq \text{diam}(kB_r), \text{ for } k \in \mathbb{Z}^+\},
$$

(40a)

where

$$
\text{diam}(X) = \max_{x_1, x_2 \in X} \bar{d}(x_1, x_2).
$$

(40b)

Observe that the RHOMBUS structuring element, depicted in Fig. 2(b), is a radial set that corresponds to the $d_i(\bullet, \bullet)$ metric defined in [19]. Similarly, the SQUARE structuring element $H$ is a radial set which corresponds to the $d_h(\bullet, \bullet)$ metric defined in [19]. Many other radial sets may be formed by different choices of the metric $\bar{d}(\bullet, \bullet)$. Notice that the class of convex and isotropic
sets are radial sets in general.\(^{11}\) In the following, we shall denote by \(B^1\) the radial set
\[
B^1 \equiv \{ x : d(x,y) \leq 1 \}
\] (41)
and we shall assume that the structuring element \(B\) is also a radial set. We shall also restrict the degradation process by assuming that
\[
\gamma_k (N_i) = \emptyset, \quad \text{for } k \geq \mu, \quad i = 1, 2, \quad \mu \in Z^+.
\] (42)
From the fact that \(\{C_{i,n} \oplus \{x_{i,n}\}, n = 1, 2, \ldots\}\) is a sequence of disconnected sets, and from (38) and (39) we see that the condition of (42) is equivalent to assuming that the primary grains \(\{C_{i,n}, n = 1, 2, \ldots\}\) are all of, at most, "size" \(\mu \) [6], [15]. We shall finally assume that
\[
\phi_{K(k)} (N_i) = N_i, \quad \text{for } 0 \leq k \leq \mu, \quad i = 1, 2, \quad \nu \in Z^+
\] (43)
for \(\nu \geq \mu\), where \(K(k) = kB \oplus B^1 \oplus H\). The condition in (43) assumes that the primary grains \(\{C_{i,n} \oplus \{x_{i,n}\}, n = 1, 2, \ldots\}\) are separated at least by the set \(\nu B \oplus B^1 \oplus H\), and that the complement \(N_i^c\) of the noise process \(N_i\) for \(i = 1, 2\), is relatively smooth with respect to the structuring element \(k\).

The noise model of (38) is a plausible model for many practical applications. This model can naturally describe binarized, discrete, colored noises, i.e., truncation of discrete noise processes with nonindependent variables [20]. The colored Gaussian and Shot noises represent two examples of colored noises that frequently appear in practical applications [20], [21].

B. Optimal Filter Parameterization

An "optimal" MF \(\hat{\Psi}_k (\bullet)\) for the morphological pattern restoration of noisy binary images may be defined to be the MF that minimizes the expected restoration error \(E[d(X, \hat{\Psi}_k (Y))]\) among the class of PMFs \(\{\Psi_k (\bullet), \lambda \in Z^+\}\), i.e.,
\[
\hat{\Psi}_k (\bullet) = \arg \min_{\lambda \in Z^+} E[d(X, \hat{\Psi}_k (Y))]
\] (44)
for every noise-free image \(X\), where \(Y\) is given by (37) and the SDDF \(d(\bullet, \bullet)\) is given by (17). In (44), \(E[d(X, \hat{\Psi}_k (Y))]\) is an average of the SDDF over all possible realizations of the noise components \(N_1\) and \(N_2\) (or, equivalently, over all measurements \(Y\)).

A direct solution to (44) requires the evaluation of the expected restoration error \(E[d(X, \hat{\Psi}_k (Y))]\) for every \(\lambda \in Z^+\), and for every noise-free image \(X\). Although this evaluation poses a salient obstacle, the difficulty may be circumvent by evaluating an "optimal" MF that minimizes an upper-bound of the expected restoration error. Let us assume that \(d_{\Psi_k} (X)\) is an upper-bound of \(d(X, \hat{\Psi}_k (Y))\), i.e.,
\[
d(X, \hat{\Psi}_k (Y)) \leq d_{\Psi_k} (X)
\] (45)
for every \(\lambda \in Z^+, Y\), and \(X\).\(^{13}\) From (44) and (45) observe that
\[
E[d(X, \hat{\Psi}_k (Y))] \leq d_{\Psi_k} (X)
\] (46)
for every \(\lambda \in Z^+\) and every noise-free image \(X\). We shall now consider an "optimal" MF \(\hat{\Psi}_k (\bullet)\) to be the MF that minimizes the upper-bound (46) of the expected restoration error among the class of PMFs \(\{\Psi_k (\bullet), \lambda \in Z^+\}\), i.e.,
\[
\hat{\Psi}_k (\bullet) = \arg \min_{\lambda \in Z^+} d_{\Psi_k} (X)
\] (47)
for every noise-free image \(X\). This is a strategy of choosing the worst possible case and trying to determine the "best" estimator for that case, which is similar to the technique of minimax estimation [22].

Let
\[
Y_1 = X - N_1
\] (48a)
and
\[
Y_2 = X \cup N_2.
\] (48b)
In the following lemma we establish a number of interesting properties that allow us to compute the bound \(d_{\Psi_k} (X)\), given by (45).

**Lemma 8:** Consider a set \(X \subseteq Z^2\).

a) Given a structuring element \(B\); i) if \(N_2 \subseteq (X \oplus H) \cap X\), then
\[
\gamma_k (Y_2) = \gamma_k (X)
\] (49)
for every \(k \geq \mu\), and ii) if \(N_1 \subseteq X^c \cup (X \oplus H)\), then
\[
\phi_k (Y_1) = \phi_k (X)
\] (50)
for every \(\mu \geq k\).

b) Given a radial structuring element \(B\), for every \(0 \leq k \leq \mu\); i) if \(N_1 \subseteq X^c \cup (X \oplus (2kB \oplus B^1))\), then
\[
\gamma_k (Y_1) = \gamma_k (X) - N_1
\] (51)
and, ii) if \(N_2 \subseteq (X \oplus (2kB \oplus B^1))^c \cup X\), then
\[
\phi_k (Y_2) = \phi_k (X) \cup N_2.
\] (52)

We shall now briefly discuss the importance of the conditions in (42) and (43). Since our problem is to derive an "optimal" MF that behaves uniformly for all possible images \(Y\), we may restrict our choice to those PMFs that result in almost perfect restoration of \(X\), for some choices of \(Y\). It is clear that such a filter should be characterized by the following two properties: a) the opening and closing operations (which are part of the AF's and the ASF's) should reduce the effect of the noise components \(N_1\) and \(N_2\) without altering the shape of image \(X\), for a sufficiently smooth image \(X\), and b) the opening (closing) operation should not alter the shape of \(X - N_1 (X \cup N_2)\), therefore allowing the image \(X - N_1 (X \cup N_2)\) to be affected only by the closing (opening) operation. Let us now assume that \(X\) is a smooth image whose pattern spectrum satisfies the condition
\[
PS_k (X) = 0, \quad \text{for } k = -m, \ldots, -1, 0, 1, \ldots, (m - 1)
\]
with \(m \geq \mu\). It is now easy to verify [see (14) and Lemma 8(a)] that a PMF \(\Psi_k (\bullet)\) satisfies property a), for \(\mu \leq \lambda \leq m\), under

\(^{11}\) In the Euclidean space the disk (ball) is the unique radial set.

\(^{12}\) A brief discussion on the need of the conditions in (42) and (43) is given in the following subsection, after Lemma 8.

\(^{13}\) For further discussion on the validity of these models, refer to [12].

\(^{14}\) According to general decision theory, \(X\) and \(Y\) are known as the parameter and the observable, respectively. Quantity \(\Psi_k (Y)\) is referred to as the estimation, whereas \(d(X, \hat{\Psi}_k (Y))\) and \(E[d(X, \hat{\Psi}_k (Y))]\) are known as the loss and risk functions, respectively [22].
condition (42), whereas a PMF $\Psi_l(\bullet)$ satisfies property b), for
\(\mu \leq \lambda \leq \nu\), under the condition of (43).

As a direct consequence of the previous discussion, we shall limit our choice to those PMF’s \(\{\Psi_l(\bullet), \lambda \in \mathbb{Z}^+\}\) that are defined for \(\mu \leq \lambda \leq \nu\), i.e., instead of solving the optimization problem (47), we shall restrict ourselves to computing PMF’s \(\Psi_l(\bullet)\) such that

\[
\Psi_l(\bullet) = \arg \min_{\mu \leq \lambda \leq \nu} d^*_n(X) \tag{53}
\]

for every noise-free image $X$. We shall discuss the solution to this problem next. Since $n_l(X) = m_l(X)$ and $N_l(X) = M_l(X)$, the results obtained for the PMF’s $n_l(\bullet)$ and $N_l(\bullet)$ will also remain valid for the PMF’s $m_l(\bullet)$ and $M_l(\bullet)$, respectively. Thus, the analysis for the ASF’s will be restricted to the PMF $N_l(\bullet)$, whereas the analysis for the AF’s will be restricted to the PMF $n_l(\bullet)$.

1) Optimal Parameterization of the AF’s: Let $B^C$ denote the set given by

\[
B^C \cap \{x\} = \{y : d(x,y) \leq \text{diam}(C)\} \tag{54}
\]

where $C = \bigcup_{i=1}^n \bigcup_{j=1}^n C_{i,j}$. We have the following proposition.

**Proposition 3:** Consider a set $X$ and the degraded image $Y$, given by (37).

a) If

\[
X'_n(\lambda) = X \ominus (2\lambda B \oplus B^t \oplus H \oplus B^C) \tag{55a}
\]

and

\[
X'_n = X \oplus (H \oplus B^C) \tag{55b}
\]

then

\[
\gamma_l(X'_n(\lambda)) \subseteq n_l(Y) \subseteq \phi_l(X'_n), \quad \text{for } \mu \leq \lambda \leq \nu. \tag{56}
\]

b) If

\[
X'_m = X \ominus (H \oplus B^C) \tag{57a}
\]

and

\[
X'_m(\lambda) = X \ominus (2\lambda B \oplus B^t \oplus H \oplus B^C) \tag{57b}
\]

then

\[
\gamma_l(X'_m(\lambda)) \subseteq m_l(Y) \subseteq \phi_l(X'_m), \quad \text{for } \mu \leq \lambda \leq \nu. \tag{58}
\]

In the following theorem we derive the optimal parameterization of the PMF $n_l(\bullet)$ by solving (53).

**Theorem 3:** Consider a set $X$ and the degraded image $Y$, given by (37).

a) For every $\mu \leq \lambda \leq \nu$

\[
d(X, n_l(Y)) \leq d^*_n(X) \tag{59a}
\]

where

\[
d^*_n(X) = \text{Card}[\phi_l(X'_n) - \gamma_l(X'_n(\lambda))]. \tag{59b}
\]

b) $n_l(\bullet) = \arg \min_{\mu \leq \lambda \leq \nu} d^*_n(X). \tag{60}$

**Proof:** a) Since $X \ominus B \subseteq X$, $X \oplus B \supseteq X$, $\gamma_l(X) \subseteq X$, $\phi_l(X) \supseteq X$ and from (55) we have

\[
X \subseteq \phi_l(X'_n) \tag{61a}
\]

and

\[
X \supseteq \gamma_l(X'_n(\lambda)). \tag{61b}
\]

From (56) and (61) we have

\[
[(X \cup n_l(Y)) - (X \cap n_l(Y))] \subseteq [(X \cup \phi_l(X'_n)) - (X \cap \gamma_l(X'_n(\lambda)))] = \phi_l(X'_n) - \gamma_l(X'_n(\lambda)) \tag{62}
\]

for $\mu \leq \lambda \leq \nu$. Observe that if $X_1 \subseteq X_2$, then $\text{Card}[X_1] \leq \text{Card}[X_2]$; therefore, from (17) and (62) we obtain (59).

b) Since $X \ominus B \subseteq X$, $\gamma_l(X) \subseteq \gamma_{l-1}(X)$, $\phi_l(X) \supseteq \phi_{l-1}(X)$ and from (55) and (59b) observe that $d^*_n(X)$ is a nondecreasing function of $\lambda$, for $\mu \leq \lambda \leq \nu$, which proves (60).

2) Optimal Parameterization of the ASF’s: We now have the following proposition.

**Proposition 4:** Consider a set $X$ and the degraded image $Y$, given by (37). If $\phi_l(Y)$ and $\phi_{l+}(Y)$ are $\lambda B$-connected sets, for $0 \leq \lambda \leq \nu$, then

\[
\gamma_l(X'_n(\lambda)) \subseteq N_l(Y) \subseteq \phi_l(X'_n), \quad \text{for } \mu \leq \lambda \leq \nu. \tag{63}
\]

In the following theorem we derive the optimal parameterization of the PMF $N_l(\bullet)$ by solving (53).

**Theorem 4:** Consider a set $X$ and the degraded image $Y$, given by (37). If $\phi_l(Y)$ and $\phi_{l+}(Y)$ are $\lambda B$-connected sets, for $0 \leq \lambda \leq \nu$, then

a) For every $\mu \leq \lambda \leq \nu$

\[
d(X, N_l(Y)) \leq d^*_n(X) \tag{64a}
\]

where

\[
d^*_n(X) = \text{Card}[\phi_l(X'_n) - \gamma_l(X'_n(\lambda))]. \tag{64b}
\]

b) $N_l(\bullet) = \arg \min_{\mu \leq \lambda \leq \nu} d^*_n(X). \tag{65}$

**Proof:** a) Since $X \ominus B \subseteq X$, $X \oplus B \supseteq X$, $\gamma_l(X) \subseteq X$, $\phi_l(X) \supseteq X$ and from (55a) and (57b) we have

\[
X \subseteq \phi_l(X'_m) \tag{66a}
\]

and

\[
X \supseteq \gamma_l(X'_m). \tag{66b}
\]

From (63) and (66) we have

\[
[(X \cup N_l(Y)) - (X \cap N_l(Y))] \subseteq [(X \cup \phi_l(X'_m)) - (X \cap \gamma_l(X'_m))] = \phi_l(X'_m) - \gamma_l(X'_m) \tag{67}
\]
for $\mu \leq \lambda \leq \nu$. Observe that if $X_1 \subseteq X_2$, then Card[$X_1$] $\leq$ Card[$X_2$]; therefore, from (17) and (67) we obtain (64).

b) Since $X \ominus B \subseteq X$, $X \ominus B \supseteq X_\gamma$, $\gamma_k(X) \subseteq \gamma_{k-1}(X)$, $\phi_k(X) \supseteq \phi_{k-1}(X)$ and from (55a), (57b), and (64b) observe that $d_{\gamma_k}(X)$ is a nondecreasing function of $\lambda$, for $\mu \leq \lambda \leq \nu$, which proves (65).

In the previous theorem we proved that the MF $N_\mu(\bullet)$ (and, equivalently, $M_\mu(\bullet)$) is the optimal MF among the class of ASF's. In this case, $\lambda = \mu$. We refer to the MF's $N_\mu(\bullet)$ and $M_\mu(\bullet)$ as the optimal ASF's. Notice that $\mu$ is an unknown associated with the degradation process. The "optimal" estimation of this parameter is an open problem. In practice, however, the parameter $\mu$ may be estimated from the pattern spectrum. For instance, consider the binary image $X$ depicted in Fig. 3(a). The degraded binary image $Y$, obtained by using (37) and (38), with $C_{i,n}$, $i = 1, 2$, formed by the overlapping of SQUARE structuring elements distributed according to a Bernoulli point process, is depicted in Fig. 3(b). In the case of the degraded image $Y$, we estimate the value of $\mu$, when filtering with a RHOMBUS structuring element, by inspecting...
the pattern spectrum \(PS_k(Y)\) of image \(Y\), depicted in Fig. 3(f). In this case the two peaks located at \(k = 1\) and \(k = -2\) are assumed to be due to the noise components, which give us the value \(\hat{\mu} = 2\). Observe that, if the SQUARE structuring elements used in the degradation process are nonoverlapping, then, from (42), we obtain \(\mu = 2\), when filtering with a RHOMBUS structuring element. In Figs. 3(c) and (d) we demonstrate the efficiency of the optimal restoration filters \(N_1(\bullet)\) and \(n_2(\bullet)\) in removing the noise degradation from the image depicted in Fig. 3(b). The corresponding pattern spectra of \(N_1(\bullet)\) and \(n_2(\bullet)\) are also depicted in Figs. 3(g) and (h), respectively. By comparing these spectra with the pattern spectrum of the noise-free image \(X\), depicted in Fig. 3(e), we immediately see the improvement obtained by the use of the ASF \(N_2(\bullet)\) over the AF \(n_2(\bullet)\). Finally, the restoration error \(d(X, \Psi(Y))\), obtained by the two filtering procedures, is depicted in Fig. 4. Observe that, in this case, the filtering by \(N_1(\bullet)\) outperforms the filtering by \(n_2(\bullet)\), and that the minimum restoration error is obtained for \(\lambda = \hat{\mu} = 2\), as it is expected.

To conclude this paper, we consider a degradation-free gray-level Matisse image\(^16\) whose binary version \(X\) is depicted in Fig. 5(a). Fig. 5(b) depicts the degraded binary image \(Y\), which is the binary version of the gray-level Matisse image corrupted by an additive, colored Gaussian noise process. The degraded binary image \(Y\) is subject to filtering by an alternating sequential filter \(\Psi(\bullet) = N_1(\bullet)\) and an alternating filter \(\Psi(\bullet) = n_2(\bullet)\), which use the RHOMBUS structuring element, to yield the "optimally" restored binary images \(\Psi(Y)\), depicted in Figs. 5(c) and (d), respectively. The value \(\hat{\mu} = 2\), when filtering with a RHOMBUS structuring element, was estimated by inspecting the pattern spectrum \(PS_k(Y)\) of image \(Y\), depicted in Fig. 6. Finally the restoration error \(d(X, \Psi(Y))\), obtained by the two filtering procedures, is depicted in Fig. 7, which demonstrates the fact that the filtering by \(N_1(\bullet)\) outperforms the filtering by \(n_2(\bullet)\), and that the minimum restoration error is obtained for \(\lambda = \hat{\mu} = 2\).

V. SUMMARY

In this paper, we have presented a theoretical analysis of morphological filters for the "optimal" restoration of noisy binary images. We have formulated the problem in its general

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These results are important on their own right in cases where computational restrictions forbid the use of alternating sequential filters.

**APPENDIX A**

**Proofs of Lemmas**

**Proof of Lemma 1:** Consider a $kB$-connected set $X$. Let us take a point $z \in X$ such that $z \notin \gamma_k(X)$. Since $\gamma_k(X) = \{z \in kB \cup \{y\} : kB \cup \{y\} \subseteq X\}$, we see that $z \notin kB \cup \{y\}$, for every $y \in X^k$ such that $kB \cup \{y\} \subseteq X$. From Definition 4 we observe that $z \notin kB \cup \{y\} \subseteq X$. Note also that if $kB \cup \{y\} \subseteq X$, then $kB \cup \{y\} \subseteq \gamma_k(X) = [\phi_k(X)]^k$, since $\phi_k(X) = \gamma_k(X)$, from which we have (with $X \to \gamma_k(X)$) that $z \notin kB \cup \{y\} \subseteq [\phi_k(X)]^k = [\gamma_k(X)]^k$, or $z \notin \gamma_k(X)$. To summarize, we have proved that, given a $kB$-connected set $X$, if there exists a point $z \in X$ such that $z \notin \gamma_k(X)$, then $z \notin \gamma_k(X).

Let us now consider a set $X$ such that $\phi_k(X)$ is a $kB$-connected set. According to the previous argument (with $X \to \phi_k(X)$), if there exists a point $z \in \phi_k(X)$ such that $z \notin \gamma_k \phi_k(X) = m_k(X)$, then $z \notin n_k \phi_k(X) = r_k(X)$ (see also (3) and (4)). Assume that $z \notin r_k(X)$; then $z \in \phi_k(X)$, since $r_k(X) \subseteq \phi_k(X)$ (see also (7a)). If $z \in m_k(X)$, then

$$m_k(X) \supseteq r_k(X).$$

(A1)

If $z \notin m_k(X)$, then from the previous argument we see that $z \notin r_k(X)$, which is a contradiction. Therefore, (A1) is always satisfied. From (7a) and (A1) we obtain

$$m_k(X) = r_k(X).$$

(A2)

Observe now that $m_k(X) = r_k(X)^c$, since $\phi_k(X)^c$ is a $kB$-connected set. From this remark we obtain $n_k(X) = n_k(X)^c$, since $s_k(X) = \phi_k(X) = \gamma_k(X)^c$ and $m_k(X) = m_k(X)^c$. Finally, from (7b) we know that $n_k(X) \subseteq r_k(X)$, which together with (A2) proves that $n_k(X) \subseteq m_k(X)$. This result, together with Property 3, completes the proof.

**Proof of Lemma 2:** From (11), and since $\gamma_{k-1}(X) \subseteq \gamma_k(X)$ and $\phi_k(X) \supseteq \phi_{k-1}(X)$, it is easy to show that (12) is satisfied if and only if

$$\gamma_k(X) = \gamma_{k+1}(X), \quad k = 0, 1, \cdots, \lambda - 1$$

(A3a)

and

$$\phi_k(X) = \phi_{k-1}(X), \quad k = 1, 2, \cdots, \lambda.$$ (A3b)

It is now easy to verify by induction that (A3) is equivalent to (A4).

**Proof of Lemma 3:** Let us consider the set $F$ given by

$$F = \{\gamma_k(\bullet), \phi_k(\bullet), k = 0, 1, \cdots, \lambda\}.$$ (A4)

Let $F$ denote the collection of transformations formed by all possible compositions of the transformations in $F$. Observe that if $L(\Psi(\bullet)) = \emptyset$ is a constraint in $C_A$, then it should be of the form

$$\Pi \Psi(X) = \Psi(x)$$

where $\Pi(\bullet) \in F$, since the constraints in $C_A$ are formed by compositions of the constraints in (15). Assume that $\Pi(\bullet)$ is an idempotent transformation (i.e., $\Pi \Pi(\bullet) = \Pi(\bullet)$). In this case the constraint (A4) generates the transformation $\Pi(\bullet)$. If we now assume that the constraint (A4) generates the transformation $\Pi(\bullet)$, then $\Pi \Pi(\bullet) = \Pi(\bullet)$, and the transformation $\Pi(\bullet)$ is idempotent. Therefore, the constraint (A4) generates the transformation $\Pi(\bullet)$ if and only if $\Pi(\bullet)$ is an idempotent transformation. From the increasing and sieving properties of opening and closing, the anti-extensivity of opening, and the extensivity of closing, we see that the collection $F$ is a size distribution (see also (9)). From Proposition 1 observe that if $\Pi(\bullet) \in F$, then $\Pi(\bullet)$ is an MF, and, therefore, it is an idempotent transformation. As a final result, the constraint of (A4) generates the MF $\Pi(\bullet)$. This completes the proof.

**Proof of Lemma 4:** i) We have

$$V(X_1 \cup X_3) - (X_1 \cap X_3) = X_3 - X_1$$

$$= (X_3 - Y) \cup (Y - X_1)$$

$$= [(X_3 \cup Y) - (X_2 \cap Y)]$$

$$\cup [(Y \cup X_3) - (Y \cap X_1)].$$

(A5)

Note that, given nonempty sets $A$ and $B$, $Card(A \cup B) = Card(A) + Card(B)$ if and only if $A \cap B = \emptyset$. Observe also that $X_2 \cap Y = Y \cup X_1$; therefore,

$$[(X_3 \cup Y) - (X_2 \cap Y)] \cap [(Y \cup X_3) - (Y \cap X_1)] = \emptyset. \quad (A6)$$

By using (17), (A5), and (A6), we can easily prove (19).

ii) We now have

$$(X_1 \cup X_3) - (X_1 \cap X_3) = X_2 - X_1$$

$$\subseteq (X_2 - Y) \cup (Y - X_1)$$

$$\subseteq [(X_2 \cup Y) - (X_2 \cap Y)]$$

$$\cup [(Y \cup X_3) - (Y \cap X_1)]$$

which, together with (17), results in (20), since $Card(A \cup B) \leq Card(A) + Card(B)$.

**Proof of Lemma 5:** Substituting (16a) in (16b) and vice versa, we obtain (see also (3) and (4))

$$m_\lambda \Psi_\lambda(Y) = \Psi_\lambda(Y)$$

(A7a)

and

$$n_\lambda \Psi_\lambda(Y) = \Psi_\lambda(Y).$$

(A7b)
Substituting (A7a) in (16a) and (A7b) in (16b), we also obtain (see also (3) and (4))

\[ r_\ell \Psi_\ell(Y) = \Psi_\ell(Y) \]  \hfill (A8a)

and

\[ s_\ell \Psi_\ell(Y) = \Psi_\ell(Y). \]  \hfill (A8b)

Observe that the MF's \( \gamma_\ell(\bullet), \phi_\ell(\bullet), m_\ell(\bullet), n_\ell(\bullet), r_\ell(\bullet) \) and \( s_\ell(\bullet) \) are generated by the constraints in (16), (A7), and (A8) (see also (2)). Moreover, by using the idempotence property of opening and closing and (2), we can easily prove that \( C_\ell^\bullet \) is a set whose elements are the constraints in (16), (A7), and (A8), since these constraints are closed under composition. This completes the proof.

**Proof of Lemma 6:** The proof is a special case of the proof of Lemma 7. \( \square \)

**Proof of Lemma 7:** 

i) The proof is obtained directly from (6) and Definitions 8 and 9.

ii) a) Let us consider the collection \( F \) of MF's \( \Psi(\bullet) \) such that: a) every \( \Psi(\bullet) \in F \) is a composition of the MF's \( m_k \), for \( 0 \leq k \leq \lambda \), and b) for every \( \Psi(\bullet) \in F \) such that \( m_k \to \Psi \), we have \( m_k \to \Psi \). Note that \( F \subseteq F^* \). From (5a) observe that \( m_k(X) \subseteq \phi_k(X) \) and \( m_k(X) \subseteq r_k(X) \). Therefore, for every MF \( \Psi(\bullet) \in F^* \), there exists an MF \( \Psi(\bullet) \in F \) such that \( \Psi(\bullet) \subseteq \Psi(\bullet) \) (see also (1)). From the sieving property of opening and closing, the anti-extensivity of opening and extensivity of closing, and from (1) and (3), we have

\[ m_k m_{k-i}(X) = m_k \gamma_{k-i} \phi_{k-i}(X) \leq m_k \phi_{k-i}(X) \]

\[ = \gamma_k \phi_{k-i}(X) = \gamma_k \phi_k(X) \]

\[ = \gamma_{k-i} \phi_{k-i} \gamma_k \phi_k(X) \]

\[ = m_k \gamma_{k-i} m_k(X) \]

for \( 0 \leq i \leq k \), or

\[ m_k m_{k-i}(X) \subseteq m_k(X) \subseteq m_{k-i} m_k(X) \]  \hfill (A9)

for \( 0 \leq i \leq k \). From (1), (2), and (A9) (with \( i = 1 \)), we also have

\[ m_k m_{k-1}(X) \supseteq m_k m_{k-1} m_{k-2}(X) \]

\[ \supseteq m_k m_{k-1} m_{k-2}(X) \]

\[ \ldots \]

\[ \supseteq m_k m_{k-1} \cdots m_{k+1} m_{k-1}(X) \]

\[ = m_k m_{k-1} \cdots m_{k+1} m_{k-1} m_{k-2}(X) \]

\[ \ldots \]

\[ = m_k(X) \]

for every \( 0 \leq k \leq \lambda \), or (see also (5))

\[ M_k(X) \subseteq m_k(X) \]  \hfill (A10)

for every \( 0 \leq k \leq \lambda \). Observe now that if \( \Psi(\bullet) \in F \), then

\[ \Psi(X) = \Pi \Psi_\ell(X) \]  \hfill (A11)

where \( \Pi \Psi(\bullet) \) and \( \Pi \Psi_\ell(\bullet) \) are MF's that are compositions of \( m_k \) filters, for \( 0 \leq k \leq \lambda \). By recursively using (A10), we can easily prove that

\[ M_k(X) \subseteq \Pi \Psi_\ell(X). \]  \hfill (A12)

By using (1) and (A9), we also have

\[ M_k(X) = m_k m_{k-1} \cdots m_{k} m_{k-1}(X) \]

\[ \subseteq \Pi \psi_\ell m_{k-1} \cdots m_{k} m_{k-1}(X) \]

\[ \subseteq \Pi \psi_\ell m_{k-1} \cdots m_{k} m_{k-1}(X) \]

\[ \ldots \]

\[ \subseteq \Pi \psi_\ell m_{k-1} \cdots m_{k} m_{k-1}(X) \]

\[ \subseteq \Pi \psi_\ell m_{k-1} \cdots m_{k} m_{k-1}(X) \]  \hfill (A13)

From (A12) and (A13) (with \( X \to \Pi \Psi_\ell(X) \)), we have

\[ M_k(X) \subseteq \Pi \psi_\ell m_{k-1} \cdots m_{k} m_{k-1}(X) \]  \hfill (A14)

From (A11) and (A14), we finally obtain that

\[ M_k(X) \subseteq \Psi(X), \]  \hfill (A15)

for all \( \Psi(\bullet) \in F \).

This completes the proof.

b) Observe that \( N_k(X) = \bar{X}_k(X) \). The proof is obtained directly from Parts i) and ii) and Property 2.

iii) From Lemma 1 we observe that \( n_k(Y) \subseteq m_k(Y) \), for \( 1 \leq k \leq \lambda \); therefore,

\[ N_k(Y) \subseteq M_k(Y). \]  \hfill (A15)

Equation (26) is a direct consequence of (A15) and Parts i) and ii).

**Proof of Lemma 8:** 

a) i) Consider the set \( N^*_k = X \cup N_k \). Observe that \( X \cup N_k \subseteq X \cup N_k \). Since \( N_k \subseteq X \cup (X \cup N_k) \), we also have that \( N^*_k \subseteq (X \cup N_k) \). And, therefore, \( X \) and \( N^*_k \) are disconnected. From (39), (42), and (48b), we finally obtain

\[ \gamma_k(Y) = \gamma_k(X \cup N_k) = \gamma_k(X \cup N^*_k) \]

\[ = \gamma_k(X) \cup \gamma_k(N^*_k) = \gamma_k(X) \]  \hfill (A16)

for \( k \geq \mu \), which proves (49). To prove (46), and (48b) also used the fact that \( \gamma_k(N^*_k) = \emptyset \), for \( k \geq \mu \), since \( \gamma_k(N^*_k) \subseteq \gamma_k(N_k) = \emptyset \), for \( k \geq \mu \) (see also (42)).

ii) From the duality relationship between the erosion and dilation (i.e., \( X \ominus B = (X \cup B)^c \)) we have that \( N_k \subseteq X^c \cup (X^c \cup N_k)^c \). From Part i) (with \( X \to X^c \) and \( N_k \to N_k \)), we have

\[ \gamma_k(X^c \cup N_k) = \gamma_k(X^c) \]  \hfill (A17)

for \( k \geq \mu \). Observe now that (see also (48a))

\[ \phi_k(Y) = \phi_k(X \ominus N_k) = \phi_k(X \ominus N_k) = [\gamma_k(X \cup N_k)]^c. \]  \hfill (A18)

From (A17) and (A18), and since \( \phi_k(X) = \gamma_k(X) \), we obtain (50).

b) i) It is known that a set \( X \) is B-open (i.e., \( \gamma_B(X) = X \)) if and only if there exists a set \( E \) such that \( X = E \ominus B \). It is also known that, if \( B = C \)-open, then \( \phi_C(X) \subseteq \phi_B(X) \), for every set \( X \). From these facts, together with the extensivity property of opening and closing (43), we have that

\[ \phi_k(N_k) = N_k, \]  \hfill (A19)

for \( 0 \leq k \leq \mu \), \( i = 1, 2 \).

From (A19) we have

\[ \gamma_k(N^*_k) = [\phi_k(N_k)]^c = N^*_k \]  \hfill (A20)
for 0 \leq k \leq \nu. Since \( \gamma_k(X_i \cap X_2) \subseteq \gamma_k(X_i) \cap \gamma_k(X_2) \), and from (48a) and (A20) we have that
\[
\gamma_k(Y_1) = \gamma_k(X - N_i) \\
\subseteq \gamma_k(X) \cap \gamma_k(N_i^c) \\
\subseteq \gamma_k(X) \cap \gamma_k(N_i) \\
\subseteq \gamma_k(X) - N_i, \quad \text{for } 0 \leq k \leq \nu. \tag{A21}
\]

Let us now denote by \( V(k) \) the set \( V(k) = \gamma_k(X) - (X \cap (2kB \oplus B^i)) \), 0 \leq k \leq \nu. Consider points \( x_1 \) and \( x_2 \) such that \( x_1 \in (X \cap (2kB \oplus B^i)) \) and \( x_2 \in [\gamma_k(X)]^c \). We can easily prove that \((2kB \oplus B^i) \oplus \{x_1\} \subseteq \gamma_k(X) \). Therefore, \( x_2 \notin (2kB \oplus B^i) \oplus \{x_1\} \). From (40), and (41) observe that \((2kB \oplus B^i) \oplus \{x_1\} = \{y : d(x_1, y) \leq \text{diam}(kB) + 1\} \), which shows that \( x \in (2kB \oplus B^i) \oplus \{x_1\} \) if and only if \( d(x, x_1) \leq \text{diam}(kB) + 1 \). This result together with the fact that \( x_2 \notin (2kB \oplus B^i) \oplus \{x_1\} \) results in
\[
\tilde{d}(x_1, x_2) \geq \text{diam}(kB) + 2 \tag{A22}
\]
for every \( x_1 \in (X \cap (2kB \oplus B^i)) \) and \( x_2 \in [\gamma_k(X)]^c \). From (A22), the fact that a set \( X \) is \( B \)-closed (i.e., \( \phi_k(X) = X \)) if and only if there is a set \( E \) such that \( X = E \ominus B \), and the fact that \( \phi_k(X) = \gamma_k(X) \) we have that, for every \( x \in V(k) \) there exists a point \( y \in Z^2 \) such that \( x \in kB \oplus \{y\} \subseteq V(k) \), which gives that
\[
\gamma_k(V(k)) = V(k). \tag{A23}
\]

Consider now a point \( x \) such that \( x \in \gamma_k(X) - N_i \). Observe that \( V(k) \subseteq \gamma_k(X) - N_i \), since \( V(k) \cap N_i = \emptyset \). In this case \( V(k) \subseteq Y_2 \), since \( \gamma_k(X) \subseteq X \). From the increasing property of opening, we obtain that \( \gamma_k(V(k)) \subseteq \gamma_k(Y_1) \), which, together with (A22), gives
\[
V(k) \subseteq \gamma_k(Y_1). \tag{A24}
\]

Equation (A24) shows that if \( x \notin V(k) \), then \( x \notin \gamma_k(Y_1) \).

Let us now take a point \( x \) such that \( x \in \gamma_k(X) - N_i \) and \( x \notin V(k) \), i.e., \( x \notin (2kB \oplus B^i) - N_i = N_i^c \cap (X \ominus (2kB \oplus B^i)) \), which shows that \( x \in N_i^c \). From (A20) we observe that \( x \in \gamma_k(N_i^c) = \gamma_k(Z^2 - N_i) \) or there exists a point \( y \in Z^2 \) such that
\[
x \in kB \oplus \{y\} \subseteq Z^2 - N_i. \tag{A25}
\]
Consider a point \( z \) such that \( z \in kB \oplus \{y\} \); hence, \( d(x, z) \leq \text{diam}(kB) \). From (A22) observe that \( z \in \gamma_k(X) \). From (A25) observe that there exists a point \( y \in Z^2 \) such that
\[
x \in kB \oplus \{y\} \subseteq \gamma_k(X) - N_i \subseteq Y_1. \tag{A26}
\]

From (A26) observe that if \( x \notin V(k) \), then \( x \in \gamma_k(Y_1) \); thus, if \( x \in \gamma_k(X) - N_i \), then \( x \in \gamma_k(Y_1) \). Finally,
\[
\gamma_k(Y_1) \supseteq \gamma_k(X) - N_i. \tag{A27}
\]

Equations (A21) and (A27) prove (51).

ii) Observe that
\[
N_i \subseteq X \cup (X^c \ominus (2kB \oplus B^i)). \tag{A28}
\]

From Part i) with \( X \to X^c \) and \( N_i \to N_i^c \) and (A28) we have
\[
\gamma_k(X^c - N_i) = \gamma_k(X^c) - N_i. \tag{A29}
\]

Observe now that (see also (48b))
\[
\phi_k(Y_2) = \phi_k(X \cup N_2) = [\gamma_k(X^c - N_2)]^c. \tag{A30}
\]

From (A29) and (A30) we finally obtain
\[
\phi_k(Y_2) = [\gamma_k(X^c) - N_2]^c = [\gamma_k(X^c)]^c \cup N_2 = \phi_k(X) \cup N_2
\]
which proves (52).

\[ \square \]

**APPENDIX B**

**Proofs of Propositions**

**Proof of Proposition 1:** Consider morphological transformations \( \Psi_1(\bullet) \) and \( \Psi_2(\bullet) \) in \( F \). According to (9a) these transformations are increasing transformations. Given \( X_1 \subseteq X_2 \), we have that
\[
\Psi_2(X_1) \subseteq \Psi_2(X_2) \tag{B1}
\]

since \( \Psi_2(\bullet) \) is increasing. By using (B1) and the increasing property of \( \Psi_1(\bullet) \), we obtain \( \Psi_1\Psi_2(X_1) \subseteq \Psi_1\Psi_2(X_2) \), which proves that the class of increasing transformations is closed under composition. In particular, the above proof implies that any morphological transformation \( \Psi(\bullet) \in F \) is increasing.

Given a set \( X \), let \( \Psi_{\text{mp}}(X) = \Gamma_{\phi_1}\Gamma_{\phi_2}\Gamma_{\phi_3}\Gamma_{\phi_4}\Gamma_{\phi_5}\Gamma_{\phi_6}\Gamma_{\phi_7}\Gamma_{\phi_8}(X) \in F \), for some indices \( i_1, i_2, \ldots, i_n \) and \( j_1, j_2, \ldots, j_n \). From (9c) it is clear that any transformation \( \Psi(\bullet) \in F \) can be represented in one of the following forms: \( \Psi_1(X) = \Psi_{\text{mp}}(X) \), \( \Psi_2(X) = \Psi_{\text{mp}}\Gamma_{\phi_1}(X) \), \( \Psi_3(X) = \Phi_2\Psi_{\text{mp}}(X) \), or \( \Psi_4(X) = \Phi_1\Psi_{\text{mp}}\Gamma_{\phi_1}(X) \), for an appropriate choice of indices \( i_1, i_2, \ldots, i_n \) and \( j_1, j_2, \ldots, j_n \). Choose \( k \) and \( l \) such that \( i_k \geq i_s \) and \( j_l \geq j_s \), respectively, for \( 1 \leq s \leq n \). From (9c) we have
\[
\Psi_1(X) = \Psi_{\text{mp}}(X) \\
= \Gamma_{\phi_1}\Gamma_{\phi_2}\Gamma_{\phi_3}\Gamma_{\phi_4}\Gamma_{\phi_5}\Gamma_{\phi_6}\Gamma_{\phi_7}\Gamma_{\phi_8}(X). \tag{B2}
\]

From (9b), (B2), and the fact that any transformation \( \Psi(\bullet) \in F \) is increasing, we have
\[
\Psi_1(X) \supseteq \Psi_1\Psi_2(X) \tag{B3}
\]

From (9c), we also have
\[
\Psi_1(X) = \Psi_{\text{mp}}(X) \\
= \Gamma_{\phi_1}\Gamma_{\phi_2}\Gamma_{\phi_3}\Gamma_{\phi_4}\Gamma_{\phi_5}\Gamma_{\phi_6}\Gamma_{\phi_7}\Gamma_{\phi_8}(X). \tag{B4}
\]

From (9b), (B4), and the fact that any transformation \( \Psi(\bullet) \in F \) is increasing, we have
\[
\Psi_1(X) \subseteq \Psi_1\Psi_2(X). \tag{B5}
\]

From (B3) and (B5), we finally obtain
\[
\Psi_1\Psi_2(X) = \Psi_1(X). \tag{B6}
\]

By following a similar argument we can show that \( \Psi_1(\bullet) \), \( \Psi_2(\bullet) \), and \( \Psi(\bullet) \) satisfy (B6). Therefore, any transformation \( \Psi(\bullet) \in F \) is idempotent. This completes the proof.

**Proof of Proposition 2:** Equations (18a) and (18b) can be directly verified from Definition 7. It is also easy to verify that
\[
[(X_1 \cup X_2) - (X_1 \cap X_2)] \subseteq [(X_1 \cup X_2) - (X_1 \cap X_2) \cup (X_1 \cup X_2) - (X_1 \cap X_2)]
\]
which, together with (17), results in (18c), since Card(A U B) ≤ Card(A) + Card(B).

Proof of Proposition 3: From (37) and (48), observe that
\[ Y_1 \subseteq Y \subseteq Y_2. \]  
(B7)

From (1) and (B7), we have
\[ n_1(Y_1) \subseteq n_1(Y) \subseteq n_1(Y_2). \]  
(B8)

Let \( V_1(\lambda) = X' \cup X \cup (2\lambda B \oplus B^t) \). From (51) observe that, for \( 0 \leq \lambda \leq \nu \), if \( N_1 \subseteq V_1(\lambda) \), then
\[ \gamma_1(Y_1) = \gamma_1(X) - N_1. \]  
(B9)

Let \( V_2(\lambda) = [\gamma_1(X)]^t \cup (\gamma_1(X) \ominus H) \). From (48a) and (50) (with \( X \rightarrow \gamma_1(X) \)), observe that if \( N_1 \subseteq V_2(\lambda) \), then
\[ \phi_1(\gamma_1(X) - N_1) = \phi_1(\gamma_1(X)) = n_1(X) \]  
(B10)

for \( \lambda \geq \mu \). Let \( V(\lambda) = X' \cup X \cup (2\lambda B \oplus B^t \ominus H) \). Since \( X \cup B \subseteq X \cup B \subseteq X \cup B \cup X' \cup X \cup B \cup X \cup B \subseteq X \cup B \cup X' \cup X \cup B \cup X \cup B \subseteq V(\lambda) \), observe that \( V(\lambda) \subseteq V(\lambda) \cup V(\lambda) \); therefore, if \( N_1 \subseteq V(\lambda) \), then \( N_1 \subseteq V_1(\lambda) \) and \( N_1 \subseteq V_2(\lambda) \). From (99) and (B10), we observe that, for \( \mu \leq \lambda \leq \nu \), if \( N_1 \subseteq V(\lambda) \), then
\[ n_1(Y_1) = \phi_1(\gamma_1(Y_1)) = \phi_1(\gamma_1(X) - N_1) = n_1(X) \]  
(B11)

Let \( N_1 = N_1^t(X) \cup N_1^t(X) \), where \( N_1 \subseteq V(X) \), and \( N_1^t(X) \) is a union of primary grains \( C_{1,n} \oplus \{x_{1,n}\} \) such that
\[ (C_{1,n} \oplus \{x_{1,n}\}) \cap V^t \neq \emptyset. \]  
(B12)

From (40) and (43), the fact that \( \phi_1(X) = \gamma_1(X) \), and the fact that \( \{C_{1,n} \oplus \{x_{1,n}\}, n = 1, 2, \ldots\} \) is a sequence of disconnected sets, observe that, for \( 0 \leq \lambda \leq \nu \),
\[ N_1^t(X) \subseteq X' \cup (X - N_1^t(X)) \ominus (2\lambda B \oplus B^t \ominus H) \subseteq (X - N_1^t(X))^t \cup (X - N_1^t(X)) \ominus (2\lambda B \oplus B^t \ominus H). \]  
(B13)

From (48a), (B11) (with \( X \rightarrow (X - N_1^t(X)) \) and \( N_1 \rightarrow N_1^t(X) \)), and (B13), we observe that, for \( \mu \leq \lambda \leq \nu \),
\[ n_1(Y_1) = n_1(X - N_1) = n_1((X - N_1^t(X)) - N_1^t(X)) = n_1(X - N_1^t(X)). \]  
(B14)

From (54), (55a), and (B12), observe that
\[ X_1^t(X) \subseteq X - N_1^t(X). \]  
(B15)

From (1), (7b), (B8), (B14), and (B15), observe that
\[ \gamma_1(X_1^t(X)) \subseteq n_1(X_1^t(X)) \subseteq n_1(X - N_1^t(X)) = n_1(Y_1) \subseteq n_1(Y) \]  
for \( \mu \leq \lambda \leq \nu \), which proves the left-hand side of (56). Now, let \( V = (X \ominus H)^t \cup X \). From (49) we observe that if \( N_2 \subseteq V \), then
\[ n_2(Y_2) = \phi_1(\gamma_1(Y_2)) = \phi_1(\gamma_1(X)) = n_1(X) \]  
(B16)

for \( \lambda \geq \mu \). Let \( N_2 = N_2^t \cup N_2^t \), where \( N_1^t \subseteq V \), and \( N_2^t \) is a union of primary grains \( C_{1,n} \oplus \{x_{1,n}\} \) such that
\[ (C_{1,n} \oplus \{x_{1,n}\}) \cap V^t \neq \emptyset. \]  
(B17)

From the fact that \( \{C_{1,n} \oplus \{x_{1,n}\}, n = 1, 2, \ldots\} \) is a sequence of disconnected sets, we have
\[ N_2^t \subseteq ((X \cup N_2^t) \ominus H)^t \cup X \subseteq ((X \cup N_2^t) \ominus H)^t \cup (X \cup N_2^t). \]  
(B18)

From (48b), (B16) (with \( X \rightarrow (X \cup N_2^t) \) and \( N_2 \rightarrow N_2^t \)), and (B18), we have
\[ n_1(Y_2) = n_1(X \cup N_2) = n_1((X \cup N_2^t) \cup N_2^t) = n_1(X \cup N_2^t) \]  
(B19)

for \( \lambda \geq \mu \). From (54), (55b), and (B17), we have
\[ X \cup N_2^t \subseteq X_1^t(X). \]  
(B20)

From (1), (7b), (B8), (B19), and (B20), observe that
\[ n_1(Y) \subseteq n_1(Y_2) = n_1(X \cup N_2) \subseteq n_1(X_1^t(X) \subseteq \phi_1(X_1^t(X)) \]  
for \( \lambda \geq \mu \), which proves the right-hand side of (56).

b) Observe that \( n_1(X) \subseteq n_1(X) \). Equation (58) is obtained directly from Part a) and Property 2.

Proof of Proposition 4: From (26), (56), and (58), we have
\[ \gamma_1(X_1^t(X)) \subseteq n_1(Y) \subseteq n_1(Y) \subseteq M_2(Y) \subseteq n_1(Y) \subseteq \phi_1(X_1^t(X)) \]  
for \( \mu \leq \lambda \leq \nu \), which proves (63).

REFERENCES


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