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On the optimality of hysteresis operators in signal processing and communication systems

Dan Schonfeld[a,*], Gary Friedman[b,*]

[a]ECE Department (M/C 154), University of Illinois at Chicago, 851 South Morgan Street–1020 SEO Chicago, IL 60607-7053, USA
[b]ECE Department, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, USA

Abstract

This paper demonstrates that hysteresis operators are the optimal estimators for a large class of estimation and decision problems. This class of problems arises whenever a cost is associated with actions needed to process changes in the estimates or decisions. A simple example related to least-squares estimation is used to illustrate and motivate the developments in this paper. Subsequently, the problem of mean-square estimation with minimal cost of actions is investigated. These results provide a justification for the use of hysteresis in many practical signal processing and communication applications.

Keywords: Hysteresis operators; Hysteresis switching; Hysteresis control; Minimal action costs; Constrained estimation; Least-squares estimation; Mean-square estimation; Cellular communications

1. Introduction

Estimation theory seeks to determine unknown parameters by minimizing a cost function that represents an estimation error. Typically, however, this error–cost function fails to capture the cost associated with the action required to process the

*Corresponding authors. Tel.: +1 312 996 5847.
E-mail addresses: ds@ece.uic.edu (D. Schonfeld), gary@ece.drexel.edu (G. Friedman).

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updated estimate. In many practical problems, however, the cost of such actions can be significant. In economics, for example, this cost stems from the cost of transactions. In electronics, power dissipation due to switching can be viewed as the cost of actions. In cellular communications, switching cost can be associated with handover between base stations.

In this paper, the problem of parameter estimation with minimal action cost is considered. Firstly, an example of least-squares estimation with minimal action-cost will be presented to illustrate the main ideas of this paper. It will be shown that optimal estimation in this problem is based on simple hysteresis operators. After that, the general notion of hysteresis will be reviewed and some important examples of hysteresis operators will be provided. Following that, the problem of minimal action-cost mean-square estimation is posed. Once again, it is demonstrated that hysteretic operators provide the optimal solution to this problem.

Hysteresis has been extensively studied in magnetism, adsorption and absorption, ferroelectric materials and other natural and man-made systems. Applications of hysteresis operators in control theory have also received some attention over the last few years [1–8]. Among these are applications where hysteresis is a property of the system under control [1,2,8]. In other cases hysteresis is employed as part of the control strategy [3–7,11–13]. Simple hysteresis operators with rectangular loops have often been used to replace thresholds in comparison operations. Schmitt trigger, for example, is discussed in almost every elementary electronics textbook [9]. Digital circuits from many manufacturers are available today with hysteresis at their inputs [10]. The use of hysteresis switching is also widespread in power electronics [11–13]. Recent work in economics indicates that hysteretic thresholds arise naturally in investment decisions [14].

It is natural to employ hysteresis for those systems which themselves exhibit hysteresis. In other applications, however, the reasons for choosing hysteresis as the desired strategy over other possible strategies have not been clearly explained or justified, except possibly on an intuitive level. The work presented in this paper is intended to shed some light on the use of hysteresis for estimation and decision problems and to show that hysteresis is not only useful, but optimal for many signal processing and communications applications.

2. Illustration and motivation

For illustration purposes, it is useful to consider a simple minimal action-cost least-squares estimation problem. Let us consider a collection of measurements $y_k(n)$ of a parameter $\theta(n)$ taken at fixed time-intervals indexed by n. The least-squares estimate $\hat{\theta}_{ls}(n)$ minimizes the square-error function given by

$$E(\theta) = \frac{1}{N}\sum_{k=1}^{N}(\theta - y_k(n))^2. \tag{1}$$

It is well known [15] that the least-squares estimate is simply the average of the measurements given by

$$\hat{\theta}_{\rm ls}(n) = \frac{1}{N}\sum_{k=1}^{N} y_k(n). \tag{2}$$

The corresponding minimum of the square-error function is denoted by $E_{\min}(n)$.

Now, suppose that changes in the optimal estimate over time incur a cost associated with the actions required to process these changes. It is possible to reduce the cost of actions by tolerating small deviations from the minimum of the square error. Let λ^2 be used to denote the maximum allowed deviation. Choosing to tolerate non-minimal errors in this way is equivalent to finding an estimate $\hat{\theta}(n)$ which minimizes the truncated version of the square error given by

$$E_t(\theta) = \begin{cases} E_{\min}(n) + \lambda^2 & \text{for } 0 \le \left(\theta - \hat{\theta}_{\rm ls}(n)\right)^2 \le \lambda^2, \\ E(\theta) & \text{otherwise.} \end{cases} \tag{3}$$

This function is illustrated in Fig. 1. It is clear from the expression above as well as from the figure that the choice of possible estimates that minimize the truncated error is not unique. This allows the possibility to select among them one which also minimizes the action cost. The action cost is characterized by the difference in the current $\hat{\theta}(n)$ and previous $\hat{\theta}(n-1)$ values of the estimate.

To demonstrate the behavior of the resulting estimator, consider the following two cases. If $\hat{\theta}(n-1)$ is within the region where the truncated error is flat, the current estimate should be the same as the previous one. If $\hat{\theta}(n-1)$ is outside the region where the truncated error is flat, the current estimate should be the boundary of the flat region closest to the previous estimate.

The relationship between the evolution of the minimal action-cost least-squares estimate $\hat{\theta}(n)$ and the least-squares estimate $\hat{\theta}_{\rm ls}(n)$ can now be easily traced. Let us

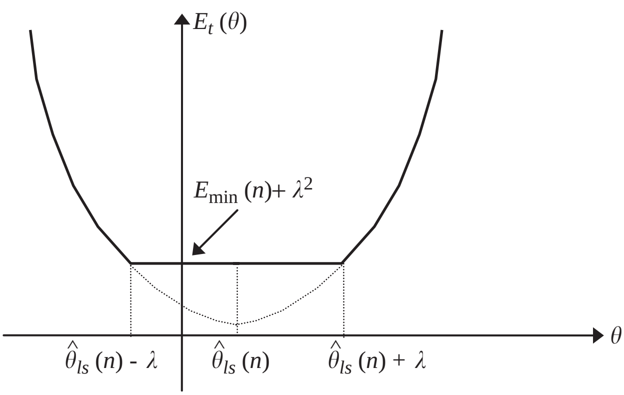

Fig. 1. Truncated square error for the minimal action-cost least-squares estimation.

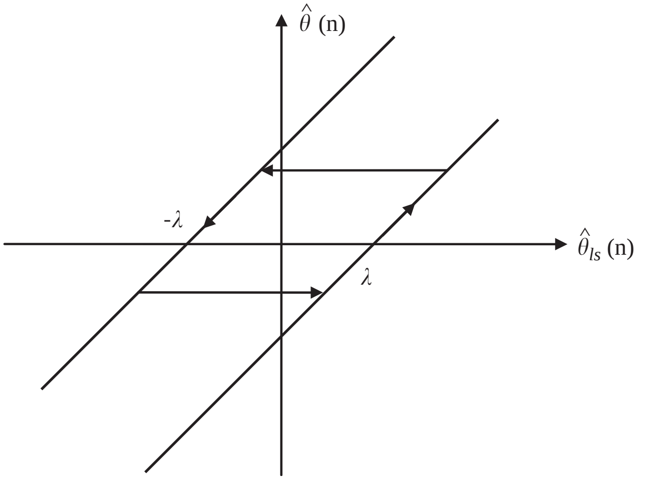

Fig. 2. The "pay or play" hysteresis estimator for the unknown scalar parameter in minimal action-cost least-squares estimation.

assume that $\hat{\theta}(n_0) = \hat{\theta}(n_0 - 1)$ for some point in time n_0. Let us now increase the least-squares estimate $\hat{\theta}_{\mathrm{ls}}(n)$. The estimate $\hat{\theta}(n)$ will remain unchanged until it coincides with the left boundary of the flat region. It will remain at the left boundary as $\hat{\theta}_{\mathrm{ls}}(n)$ increases. Therefore, further increase in $\hat{\theta}_{\mathrm{ls}}(n)$ will cause an equal increase in $\hat{\theta}(n)$. Now, assume that at some time $\hat{\theta}_{\mathrm{ls}}(n)$ reverses its direction of variation. The current estimate will again remain unchanged until it coincides with the right boundary of the flat region. Any subsequent decrease in $\hat{\theta}_{\mathrm{ls}}(n)$, will cause an equal decrease in the current estimate. This dependence of the current estimate on the least-squares estimate is illustrated in Fig. 2 where the arrows indicate the directions of variation.

The relation between the variables in Fig. 2 is hysteretic. In fact, this type of hysteresis operator is known as the play operator [16]. In connection with the above problem it may be better to call it a "pay or play" operator. Indeed, whenever the estimate in this problem deviates slightly from the least-squares estimate, some "play" can occur with no action-cost to be paid as a result of small changes in measurements. When the estimate in this problem tends to deviate substantially from the least-squares estimate, there will be action-cost to "pay" in keeping the errors sufficiently small.

3. Hysteresis

The phenomenon of hysteresis has been observed for a long time and discussed in depth in a series of books [16–19]. This phenomenon can be traced back to the existence of multiple stable states of a system. The output of this system, therefore, depends not only on the current input, but also on the history which determines the state. The reliance on history implies the existence of memory in hysteretic systems. The memory exhibited by hysteretic systems, however, is distinct from dynamic memory. Hysteretic memory is independent of the rate of input variations. Dynamic memory, on the other hand, is attributed solely to the dependence on the rate of the input variations.[1]

Let us consider *hysteresis* to be a mapping from an input sequence $\{\bar{y}_n\}$ to an output sequence $\{\bar{f}_n\}$ such that any number of consecutively repeating input values produces the same number of consecutively repeating output values. One of the ramifications of this definition of hysteresis is that it precludes transient behavior and, thus, implies independence of the rate of the input variations. This definition of hysteresis is consistent with and embodies a wide class of hysteresis models [16–19].

An important class of hysteresis operators can be described by recursive projections. Consider a collection of bounded sets $\Phi(\bar{y}_n) \subset \Re^k$ for $\bar{y}_n \in \Re^m$ and let the function $D(\bar{x}, \bar{v})$ be a measure of the distance between $\bar{x}, \bar{v} \in \Re^k$. The output of the hysteresis operator given input $\bar{y}_n$ is specified by the projection of the previous output onto $\Phi(\bar{y}_n)$, [2] i.e.,

$$\hat{P}_n = \arg \min_{\bar{v} \in \Phi(\bar{y}_n)} D(\bar{v}, \hat{P}_{n-1}). \tag{4}$$

It is not hard to demonstrate that the above operator is a hysteresis operator: Firstly, the above expression can be viewed as a mapping from an input sequence $\{\bar{y}_n\}$ to an output sequence $\{\hat{P}_n\}$. Secondly, consecutive repetitions of the input produce consecutive repetitions of the output due to the fact that projections are invariant to repeated iterations. Note that this mapping exhibits memory because the current value of the output depends not only on the current value of the input, but also on the previous value of the output.

An important example of the recursive projection hysteresis operator is a recursive spherical projection obtained when $\Phi(\bar{y}_n)$ is a sphere of some radius λ centered at $\bar{y}_n$. In this case the expression for the operator output is given by

$$\hat{P}_n = \begin{cases} \bar{y}_n - \lambda \frac{\bar{y}_n - \hat{P}_{n-1}}{|\bar{y}_n - \hat{P}_{n-1}|}, & |\bar{y}_n - \hat{P}_{n-1}| \geq \lambda, \\ \hat{P}_{n-1} & \text{otherwise.} \end{cases} \tag{5}$$

The output of the recursive projection operator can, by definition, take on any value in $\Re^k$. If, however, the output is restricted to be in some $\Theta \subset \Re^k$, the hysteresis operator can be defined by the projection of the output $\hat{P}_n$ onto Θ as follows:

$$\hat{\gamma}_n = \arg \min_{\bar{u} \in \Theta} D(\hat{P}_n, \bar{u}). \tag{6}$$

[1] Some authors refer to hysteretic systems as systems exhibiting both hysteretic and dynamic memory [20].

[2] Although for simplicity the Euclidean space has been considered, the discussion is valid for arbitrary metric spaces.

In the particular case when $\Theta = \{1, 0\}$ and the input is a scalar quantity, the hysteresis operator with output $\hat{\gamma}_n$ is the most elementary hysteresis operator and can be represented by a rectangular loop on its input–output plane.

4. Minimal action-cost mean-square estimation

The cost function employed in mean-square estimation is typically used to account for estimation errors. The justification for this approach is the fact that the risk associated with estimation errors is often the only consideration. This formulation, however, fails to account for the cost of action required to change the estimate. In many applications, it may be desired to maintain the current estimate even if it results in a larger estimation error. We propose to balance the minimization of estimation errors and cost of actions in a constrained optimization framework.

The costs of action represent the cost incurred by a change in the system parameters.

Definition 1. The *action cost* $c(\bar{\theta}_1, \bar{\theta}_2)$ of transition from parameter $\bar{\theta}_1$ to $\bar{\theta}_2$ is given by $\left\|\bar{\theta}_1 - \bar{\theta}_2\right\|$.

The *mean-squared estimate* $\hat{\theta}_{\mathrm{MS},n}$ of the parameter $\bar{x}_n$ is defined as

$$\hat{\theta}_{\mathrm{MS},n} = \arg\min_{\bar{\theta}} E\left[\left\|\bar{\theta}_n - \bar{x}_n\right\|^2\right]. \tag{7}$$

From the fundamental theorem of estimation theory [15] it can be easily shown that the mean-squared estimate $\hat{\theta}_{\mathrm{MS},n}$ is given by its conditional mean, i.e.,

$$\hat{\theta}_{\mathrm{MS},n} = E\left[\bar{x}_n | \bar{y}_n\right]. \tag{8}$$

In many practical problems there exists an uncertainty in the assumed probability model. Allowing for deviation from the minimal error estimate is particularly appealing in the presence of model uncertainty. The error-tolerance parameter λ^2 can be interpreted as a parameter characterizing the model uncertainty. The class $\Psi(\bar{y}_n)$ representing a set of estimates in an uncertainty region around the mean-square estimate $\hat{\theta}_{\mathrm{MS},n}$ is given by

$$\Psi_n(\bar{y}_n) = \left\{\bar{\theta}\middle|\left\|\bar{\theta} - E\left[\bar{x}_n | \bar{y}_n\right]\right\|^2 \leq \lambda^2\right\}. \tag{9}$$

The *minimal action-cost mean-square* estimate $\hat{\theta}_n$ is defined as the estimate within the uncertainty region $\Psi(\bar{y}_n)$ which minimizes the cost of action, i.e.,

$$\hat{\theta}_n = \arg\min_{\bar{\theta}} \left\|\bar{\theta} - \hat{\theta}_{n-1}\right\|, \tag{10}$$

such that

$$\left\|\bar{\theta} - E\left[\bar{x}_n | \bar{y}_n\right]\right\|^2 \leq \lambda^2. \tag{11}$$

In the following proposition, we derive the minimal action-cost mean-square estimate $\hat{\theta}_n$. The proof is given in the appendix.

Proposition 1. The minimal action-cost mean-square estimate $\hat{\theta}_n$ is given by

$$\hat{\theta}_n = \begin{cases} E\left[\bar{x}_n \middle| \bar{y}_n\right] - \lambda \frac{E\left[\bar{x}_n \middle| \bar{y}_n\right] - \hat{\theta}_{n-1}}{\left\| E\left[\bar{x}_n \middle| \bar{y}_n\right] - \hat{\theta}_{n-1} \right\|}, & \left| E\left[\bar{x}_n \middle| \bar{y}_n\right] - \hat{\theta}_{n-1} \right| \geq \lambda, \\ \bar{\theta}_{n-1}, & \left| E\left[\bar{x}_n \middle| \bar{y}_n\right] - \hat{\theta}_{n-1} \right| \leq \lambda. \end{cases} \quad (12)$$

It is clear that the uncertainty region $\Psi(\bar{y}_n)$ of estimates which deviate from the mean-square error estimate $\hat{\theta}_{\mathrm{MS},n}$ by no more than λ^2 is a sphere with radius λ centered at $E\left[\bar{x}_n \middle| \bar{y}_n\right]$. From the discussion in Section 3, we observe that the minimal action-cost mean-square estimate is a recursive spherical projection hysteresis operator. Whenever the previous estimate $\hat{\theta}_{n-1}$ is within this sphere, the current estimate should be set equal to the previous estimate because that will result in zero action-cost while keeping the error within the desired bounds. Whenever the sphere does not contain the previous estimate, the new estimate will be located on the boundary of the sphere at the point closest to the previous estimate. This will again result in the lowest action-cost while keeping the error within its bounds.

The behavior of the estimator above may be difficult to visualize, except possibly in the scalar case shown in Fig. 3. This hysteresis operator is identical to the one obtained in Section 2 and shown in Fig. 2, except, ofcourse, for the variable used as the input to this operator.

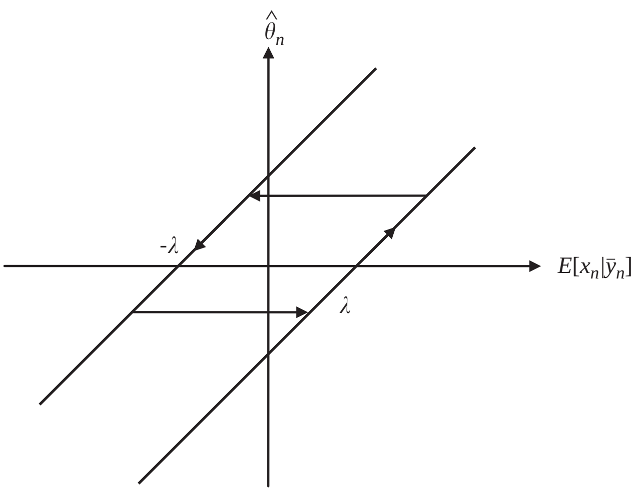

Fig. 3. The ‘‘pay or play’’ hysteresis estimator for the unknown scalar parameter in minimal action-cost mean-square estimation.

5. Cellular communications

We shall now illustrate an example of the role hysteresis plays in signal processing and communications. The particular example that we will outline is from cellular communications. Specifically, we will show that hysteresis is commonly used to minimize error estimation and action-costs in wireless communications.

In wireless communications, an overlapping network of cellular regions covers the entire communication zone. Communication by a mobile device is provided through the base station in its cellular region. Mobile devices migrating from a cellular region to its adjacent cells must switch between the corresponding base stations. This form of cellular switching is known as *call handoff* and is illustrated in Fig. 4.

A simple approach to call handoff in cellular communications is to switch to the base station whose signal strength at the mobile receiver is greater. This practice, however, has been shown to result in *ping-ponging*–wild switching between links to base stations when the mobile device is between the base stations. Moreover, this approach to call handoff results in too many handoffs [21].

Call handoffs incur a cost for switching the link of mobile devices to a new base station. This cost is manifested in the computational burden imposed on the base stations and the mobile switching office connecting the base stations to the communication network. It is therefore desired to minimize the number of handoffs in cellular networks. This goal is achieved by using hysteresis to prevent handoffs when mobile devices migrate between neighboring cells [21,22].

When a mobile device is moving from one cell to its neighboring cell, handoff between the base stations is delayed beyond the moment that the received signal strengths of the base stations has changed. Switching between the base stations is only permitted when the received signal power of the new base station exceeds the power of the original base station by a specified threshold. This operation is precisely a hysteresis switching operation similar to the operation depicted in Fig. 3. It is commonly used in many wireless communication standards (e.g., GSM).

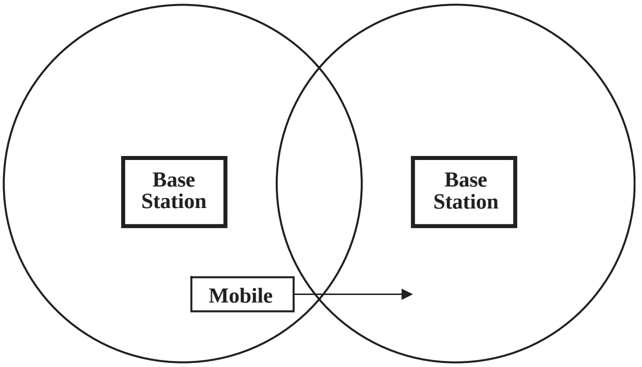

Fig. 4. Migration of a mobile device between adjacent cells.

6. Summary

In this paper, the problem of minimizing the cost of actions in the course of finding an estimate or making a decision has been considered. A simple problem of minimizing action-cost in the context of least-squares estimation has been used to illustrate and motivate the general ideas presented in this paper. It has then been shown that recursive projection hysteresis operator is optimal in reducing the cost of actions when the error risk is constrained near its minimum in mean-square estimation. The observation that hysteresis provides the minimum action-cost in the context of parameter estimation can be easily extended to other estimation criteria: e.g., Bayesian estimation and hypothesis testing.

As we have demonstrated in this paper, action-costs play an important role in signal processing and communications systems. An appreciation of the role of hysteresis may have a profound impact on signal processing and communication applications in which action-costs arise naturally. The results presented in this paper can be used to provide an explanation of the present use of hysteretic algorithms in existing signal processing and communication systems. The effect of action-costs has been indirectly incorporated in some signal processing and communication applications through the use of hysteresis switching (e.g., call handoff in cellular communication systems). This use of switching in signal processing and communications has been motivated on an intuitive level and not explicitly related to the notion of action costs. In many signal processing and communication systems, however, the implicit presence of action costs has generally been ignored. Utilization of hysteretic operators in these applications will likely increase once the notion of action costs is incorporated. We hope that this paper will help motivate the broad use of hysteretic algorithms across many future signal processing and communication systems.

Appendix. Proof of proposition

The *minimal action-cost mean-square* estimate $\hat{\theta}_n$ is given by

$$\hat{\theta}_n = \arg\min_{\bar{\theta}} \left\| \bar{\theta} - \hat{\theta}_{n-1} \right\|, \tag{A.1}$$

such that

$$\|\bar{\theta} - E[\bar{x}_n|\bar{y}_n]\|^2 \leq \lambda^2. \tag{A.2}$$

It is clear that when $\left|E[\bar{x}_n|\bar{y}_n] - \hat{\theta}_{n-1}\right| \leq \lambda$, then $\hat{\theta}_n = \hat{\theta}_{n-1}$.

Let us now assume that $\left|E[\bar{x}_n|\bar{y}_n] - \hat{\theta}_{n-1}\right| \geq \lambda$. From (A.1) and (A.2) we use Lagrange multipliers to obtain

$$\hat{\theta}_n = \arg\min_{\bar{\theta},\mu} \left\| \bar{\theta} - \hat{\theta}_{n-1} \right\|^2 + \mu\left[\|\bar{\theta} - E[\bar{x}_n|\bar{y}_n]\|^2 - \lambda^2 \right]. \tag{A.3}$$

Computing the derivative of (A.3) with respect to $\bar{\theta}$ and setting to zero we obtain

$$\hat{\theta}_n = \frac{1}{1+\mu}\left[\hat{\theta}_{n-1} + \mu E\left[\bar{x}_n \middle| \bar{y}_n\right]\right]. \tag{A.4}$$

From (A.2) and (A.4) we have

$$\left\| \frac{1}{1+\mu}\left[\hat{\theta}_{n-1} + \mu E\left[\bar{x}_n \middle| \bar{y}_n\right]\right] - E\left[\bar{x}_n \middle| \bar{y}_n\right] \right\|^2 \leq \lambda^2. \tag{A.5}$$

Therefore, we observe that

$$\mu = \frac{\left\| \hat{\theta}_{n-1} + \mu E\left[\bar{x}_n \middle| \bar{y}_n\right] \right\|}{\lambda} - 1. \tag{A.6}$$

Now, using (A.6) to evaluate (A.4) we obtain

$$\hat{\theta}_n = E\left[\bar{x}_n \middle| \bar{y}_n\right] - \lambda \frac{E\left[\bar{x}_n \middle| \bar{y}_n\right] - \hat{\theta}_{n-1}}{\left\| E\left[\bar{x}_n \middle| \bar{y}_n\right] - \hat{\theta}_{n-1} \right\|}. \tag{A.7}$$

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