Morphological Representation of Discrete and Binary Images

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***Abstract*—We present a general theory for the morphological representation of discrete and binary images. The basis of this theory relies upon the generation of a set of nonoverlapping segments of an image via repeated erosions and set transformations, which in turn produces a decomposition that guarantees exact reconstruction. We introduce a general morphological representation and we study its properties. We also investigate the relationship between the proposed representation and some existing shape analysis tools (e.g., discrete size transform, pattern spectrum, skeletons), thus introducing the representation as the basis of a unified theory for geometrical image analysis. Particular cases of the general representation scheme are shown to yield a number of useful image decompositions which are directly related to various forms of morphological skeletons. We study the relationship between the representation and the various forms of morphological skeletons. As a result of this study we develop a unified theory for the mathematical description of the morphological skeleton decomposition of discrete and binary images.**

I. Introduction

In many image analysis applications there is a need to develop an image representation scheme which contains various important aspects of the image in a compressed form. Particularly, by developing the necessary mathematical tools, we are usually able to transform an image into a set of simpler images which contain sufficient information about the shape, size, orientation, and geometry of the image under consideration. The representation scheme can then be effectively used for the design of automated image analysis and computer vision systems.

One important image representation scheme, for continuous, binary images, is the medial axis transform, or the symmetric axis transform. This transform has been originally proposed by Blum [1], [2] and is known as the skeleton [3]–[5]. The medial axis transform was initially defined to be the locus of the points of intersection of waves which are sent from the image boundary towards its center, at a uniform speed. The reconstruction of the original image was shown to be possible by a backward propagation of the waveforms, provided that the arrival times of the waves on the medial axis are known. In a subsequent development of the theory, Blum [2] proved that the medial axis of an image is the locus of the centers of the maximal disks that can be inscribed inside the image. Additional results derived by Pfaltz and Rosenfeld [3], Calabi and Hartnett [4], Montanari [5], Mott-Smith [6], Matheron [7], [8], Meyer [9], and others provided the mathematical framework for a systematic study of the skeleton and its properties. As a result of this effort, the skeleton has been established as a useful image analysis tool [10]–[14].

By using the principles of mathematical morphology [15], [16], Lantuejoul was able to prove that the skeleton can be derived by using elementary morphological operations (i.e., erosions and dilations) [16]. As a result of Lantuejoul's approach, the development of the morphological skeleton resulted in a mathematically rigorous understanding of the skeleton and its properties. In an attempt to digitize the morphological skeleton, Serra [16] proposed the discrete morphological skeleton transform of a discrete and binary image which is sampled on a hexagonal grid and uses a hexagonal structuring element. Maragos and Schafer [17] extended this result to binary images, which are sampled on a rectangular grid, by using an arbitrary discrete structuring element, whereas, Maragos [18] proposed the generalized discrete morphological skeleton transform which can be applied to any discrete and binary image by using a sequence of arbitrary structuring elements. It can be proved that the information included in the skeleton of an image is sufficient to exactly reconstruct the image, independently of the type of the structuring elements used. Various important properties of the discrete morphological skeleton transform appear in [17], [18].

An important subject in the skeleton decomposition of discrete and binary images is the issue of globally and locally minimal skeletons. In many applications of interest (e.g., image coding), it is desirable to develop an image decomposition which contains the minimum possible number of points. One way to accomplish this task is through globally minimal skeletons. The globally minimal skeleton of an image is defined as the skeleton whose points are sufficient for the exact reconstruction of the original image, but the removal of any one of these points will result only in a partial reconstruction. Unfortunately, the computation of a globally minimal morphological skeleton is a very difficult task in practice, and the resulting skeleton may violate many fundamental skeleton properties. The locally minimal skeleton is a skeleton that

Manuscript received September 11, 1988; revised June 18, 1990. This work was supported in part by a joint development program with the Federal Systems Division of IBM.

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IEEE Log Number 9143800.

contains fewer points than the discrete morphological skeleton transform and it satisfies part or all of the skeleton properties. Two interesting algorithms have been developed in [17] for the computation of globally and locally minimal skeletons. Recently, Maragos [19], [20] introduced the reduced morphological skeleton transform. This skeleton is proved to be a locally minimal skeleton that contains fewer points than the discrete morphological skeleton transform, while it preserves many of the skeleton properties.

The connection between the discrete morphological skeleton transform and the medial axis transform is studied in [17]. Under the assumption of convexity of the structuring element under consideration, the discrete morphological skeleton transform is proved to be equivalent to a discrete version of the medial axis transform (see [17, theorem 1]). Unfortunately, the reduced morphological skeleton transform does not necessarily satisfy this equivalence property [20], but it turns out to be fully compatible to the pattern spectrum, a useful shape-size tool for image analysis [19], [20].

Many interesting questions, concerning the use and nature of morphological skeletons, may be raised at this point. Are the various skeletons presented in [16]-[20] the only possible morphological image representation schemes, or there exists a more general class of morphological representations which satisfies desirable image decomposition properties (e.g., exact reconstruction) and whose special cases are the previously discussed skeletons? Is the reduced morphological skeleton transform the only possible morphological skeleton, within a general class of morphological representations, which is fully compatible to the pattern spectrum? Is the discrete morphological skeleton transform the only possible morphological skeleton, within the same class of morphological representations, that is equivalent to a discrete version of the medial axis transform? It is clear that if we develop a general class of morphological representations of discrete and binary images, we may be able to answer these questions. Additionally, we may be able to study various possibilities for morphological image representation in a more abstract form than in [16]-[20].

In this paper we present a general class of morphological image representations and we study its properties. A theory is developed for the morphological representation of discrete and binary images. This theory allows us to study the skeleton representation of images in a general form and to derive useful results and properties for its behavior. In Section II a general morphological image representation is introduced. Several properties are studied and conditions are derived for the invertibility, translation invariance, and efficiency of the representation. The relationship between the proposed representation and morphological openings is discussed and its link to the pattern spectrum is established. Some examples illustrate special cases of morphological image representation. In Section III, the subject of morphological skeleton representation of discrete and binary images is studied in connection to the proposed morphological image representation. A discrete version of the medial axis transform is introduced formally, and conditions are derived for its relationship with the morphological image representation. Finally, in Section IV we conclude our paper with a brief summary of our results.

II. Morphological Representation of Discrete and Binary Images

In this section we present a general theory for the morphological representation of discrete and binary images. The basis for this theory relies upon the generation of a set of nonoverlapping segments of the image via repeated erosions and set transformations, which in turn produces a decomposition that guarantees the exact reconstruction of the original image. Particular cases of the general representation scheme will be shown to yield a number of useful image representations which are directly related to various forms of digital morphological skeletons.

A. Morphological Image Representation

A fundamental image decomposition procedure is to define the sets $R_{seg,n}(X)$ by

$$R_{seg,n}(X) = X \ominus A(n) - X \ominus A(n+1), \quad \text{for } n = 0, 1, \cdots, N \tag{1}$$

where X is a set representing the original discrete and binary image

$$A(0) = \{(0, 0)\} \tag{2a}$$

$$A(n+1) = A(n) \oplus B(n), \quad \text{for } n = 0, 1, \cdots, N-1 \tag{2b}$$

$N = \max_n \{n; X \ominus A(n) \neq \varnothing\}$, and $\{B(n), n = 0, 1, \cdots, N-1\}$ is a sequence of structuring elements such that $(0, 0) \in B(n)$ and $B(n) \neq \{(0, 0)\}$, for all n. In our work $A \ominus B$, $A \oplus B$ and $A - B$ denotes the erosion, dilation, and set difference, respectively, between the sets A and B.[1] From (2) it is easily seen that

$$X \ominus A(n+1) = [X \ominus A(n)] \ominus B(n) \subseteq X \ominus A(n) \tag{3}$$

for $n = 0, 1, \cdots, N-1$. By using (1) and (3) we obtain that

$$R_{seg,n_1}(X) \cap R_{seg,n_2}(X) = \varnothing, \quad \text{for } n_1 \neq n_2. \tag{4}$$

Observe that

$$R_{seg,n}(X) \subseteq X, \quad \text{for } n = 0, 1, \cdots, N \tag{5}$$

and

$$X = \bigcup_{n=0}^{N} R_{seg,n}(X);$$

therefore, the image X is decomposed into a sequence $\{R_{seg,n}(X), n = 0, 1, \cdots, N\}$ of $N+1$ nonoverlapping segments which guarantee exact reconstruction.

[1]Some useful definitions and properties of morphological transformations can be found in [15], [16], [21].

Although the previous decomposition satisfies our requirements, it does not result in a useful image representation, since no redundant information is removed from the image X under consideration. If we now define a sequence of general set transformations $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$, such that

$$X \ominus A(n+1) \subseteq \Psi_n[X \ominus A(n+1)] \subseteq X \ominus A(n), \quad \text{for } n = 0, 1, \cdots, N \qquad (6)$$

and for every image $X \subseteq Z^2$, and if we define subsets $R_n(X)$ by

$$R_n(X) = X \ominus A(n) - \Psi_n[X \ominus A(n+1)], \quad \text{for } n = 0, 1, \cdots, N \qquad (7)$$

then clearly (see (1), (6) and (7))

$$R_n(X) \subseteq R_{\text{seg},n}(X), \qquad \text{for } n = 0, 1, \cdots, N. \qquad (8)$$

In this case,

$$\text{Card}\,[R_n(X)] \leq \text{Card}\,[R_{\text{seg},n}(X)], \quad \text{for } n = 0, 1, \cdots, N$$

where Card $[X]$ denotes the cardinality of set X; i.e., the total number of elements in X. The motivation for restriction (6) is to obtain a collection of disjoint subsets $\{R_n(X), n = 0, 1, \cdots, N\}$ (as it will be shown in Proposition 2(a)) that satisfy inequality (8). If the sequence $\{R_n(X), n = 0, 1, \cdots, N\}$ is also invertible, i.e., if there exists a sequence of set transformations $\{G_n[\bullet], n = 0, 1, \cdots, N\}$, such that

$$X = \bigcup_{n=0}^{N} G_n[R_n(X)]$$

then the image decomposition in terms of $\{R_n(X), n = 0, 1, \cdots, N\}$ may provide an efficient image representation scheme. It is the purpose of our work to investigate the possible choices for the sequence of set transformations $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ and to study the resulting image representations that correspond to these choices.

Definition 1: Consider a sequence $\{B(n), n = 0, 1, \cdots, N-1\}$ of structuring elements such that $(0, 0) \in B(n)$ and $B(n) \neq \{(0, 0)\}$, for all n, a sequence $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ of set transformations constrained by (6) and by $\Psi_n[\varnothing] = \varnothing$, for $n = 0, 1, \cdots, N$, and a discrete, binary image $X \subseteq Z^2$. The morphological image representation $R(X)$ of image X is defined by

$$R(X) = \{R_0(X), R_1(X), \cdots, R_N(X)\} \qquad (9)$$

where $R_n(X)$ is the morphological image representation subset of order n, given by (7), with $A(n)$ as defined in (2).

As we shall see in the following, $R(X)$ is an important image representation which decomposes image X into a collection $\{R_n(X), n = 0, 1, \cdots, N\}$ of $N+1$ disjoint subsets. This set of subsets is proved to contain sufficient information to uniquely represent the original image X. We next present a theorem which defines a restriction on the choices of the sequence $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ as a direct consequence of constraint (6), and we establish the invertibility of $R(X)$ under this restriction.[2]

Theorem: If the sequence of set transformations $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ satisfies constraint (6), for every image $X \subseteq Z^2$, then

$$X \ominus A(n+1) \subseteq \Psi_n[X \ominus A(n+1)] \subseteq [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \qquad (10)$$

for $n = 0, 1, \cdots, N$. Moreover, $R(X)$ is invertible and

$$X = R^{-1}[R(X)] = \bigcup_{n=0}^{N} [R_n(X) \oplus A(n)]. \qquad (11)$$

Proof: Since constraint (6) must be satisfied for every image $X \subseteq Z^2$, we can choose $X \rightarrow X \bigcirc A(n+1)$, and, therefore, we obtain

$$[X \bigcirc A(n+1)] \ominus A(n+1) \subseteq \Psi_n[[X \bigcirc A(n+1)] \ominus A(n+1)] \bullet [X \bigcirc A(n+1)] \ominus A(n) \qquad (12)$$

for $n = 0, 1, \cdots, N$. Observe that

$$[X \bigcirc A(n+1)] \ominus A(n+1) = [X \ominus A(n+1)] \bullet A(n+1)$$

and since $X \ominus A(n+1)$ is $A(n+1)$-closed, we have

$$[X \bigcirc A(n+1)] \ominus A(n+1) = X \ominus A(n+1). \qquad (13)$$

Also,

$$\begin{aligned} [X \bigcirc A(n+1)] \ominus A(n) &= [[X \ominus A(n+1)] \oplus A(n+1)] \ominus A(n) \\ &= [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n). \end{aligned} \qquad (14)$$

If we substitute (13) and (14) into (12) we obtain (10).

From (7) we now have

$$X \ominus A(n) = R_n(X) \cup \Psi_n[X \ominus A(n+1)] \qquad (15)$$

where $\Psi_n[\bullet]$ satisfies (10). From (10) and (15), and since $X \oplus A(n)$ is $A(n)$-open, for every X, we have

$$\begin{aligned} X \bigcirc A(n) &= [R_n(X) \cup \Psi_n[X \ominus A(n+1)]] \oplus A(n) \\ &= [R_n(X) \oplus A(n)] \\ &\quad \cup [\Psi_n[X \ominus A(n+1)] \oplus A(n)] \\ &\subseteq [R_n(X) \oplus A(n)] \\ &\quad \cup [[[X \ominus A(n)] \bigcirc B(n)] \bullet A(n)] \oplus A(n) \\ &= [R_n(X) \oplus A(n)] \\ &\quad \cup [[[X \ominus A(n)] \bigcirc B(n)] \oplus A(n)] \bigcirc A(n) \\ &= [R_n(X) \oplus A(n)] \cup [[X \ominus A(n)] \bigcirc B(n)] \\ &\quad \oplus A(n) \\ &= [R_n(X) \oplus A(n)] \cup [X \bigcirc A(n+1)]. \end{aligned} \qquad (16)$$

[2]In the following, $A \bigcirc B$ and $A \bullet B$ denote the opening and closing, respectively, of a set A by the structuring element B.

Let

$$\Phi_n = X \bigcirc A(n). \tag{17}$$

From (16) and (17) we have

$$\Phi_n \subseteq [R_n(X) \oplus A(n)] \cup \Phi_{n+1}. \tag{18}$$

Observe that $\Phi_{N+1} = \varnothing$ and $\Phi_N = R_N(X) \oplus A(N)$. By iterating (18), for $n = k, k + 1, \cdots, N$, we obtain

$$\Phi_k \subseteq \bigcup_{n=k}^{N} [R_n(X) \oplus A(n)]. \tag{19}$$

From (7) we know that $R_n(X) \subseteq X \ominus A(n)$; thus, $R_n(X) \oplus A(n) \subseteq X \bigcirc A(n)$. In this case,

$$R_n(X) \oplus A(n) \subseteq \Phi_n, \qquad \text{for } n = 0, 1, \cdots, N. \tag{20}$$

It is clear that $A(n + 1)$ is $A(n)$-open; thus $X \bigcirc A(n + 1) \subseteq X \bigcirc A(n)$, which results in

$$\Phi_{n_2} \subseteq \Phi_{n_1}, \qquad \text{for } n_1 \leq n_2. \tag{21}$$

From (20) and (21) we have

$$\bigcup_{n=k}^{N} [R_n(X) \oplus A(n)] \subseteq \Phi_k. \tag{22}$$

Equations (19) and (22) prove that

$$\Phi_k = \bigcup_{n=k}^{N} [R_n(X) \oplus A(n)]. \tag{23}$$

Observe that $\Phi_0 = X \bigcirc A(0) = X \bigcirc \{(0, 0)\} = X$, which, together with (23), proves (11). □

From (14) observe that $[[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \subseteq X \ominus A(n)$; therefore, $[[X \ominus A(n)] \bigcirc B(n)] \bullet A(n)$ is, in general, a tighter upper bound for $\Psi_n[\bullet]$ than $X \ominus A(n)$.

In the remainder of this paper we shall restrict $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ to satisfy (10), thereby allowing for a representation $R(X)$ which permits the exact reconstruction of X. There exist many choices for $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ which satisfy (10). We shall explore some of these choices later; however, in the remainder of this section we shall investigate some general properties of $R(X)$. Special cases of some of these properties and their corresponding proofs can be found in [17], [18].

First, we shall show that $R(X)$ is a translation invariant transformation, when $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ is a sequence of translation invariant mappings.

Proposition 1: If $\Psi_n[X \oplus \{z\}] = \Psi_n[X] \oplus \{z\}$, for $n = 0, 1, \cdots, N$, then

$$R(X \oplus \{z\}) = R(X) \oplus \{z\}$$

for every $z \in Z^2$.

Proof: The proof is a direct consequence of the translation invariance of $\Psi_n[\bullet]$, the translation invariance of the erosion, and the fact that $(A \oplus \{z\}) - (B \oplus \{z\}) = (A - B) \oplus \{z\}$. □

Proposition 2:

$$\text{a) } R_{n_1}(X) \cap R_{n_2}(X) (X) = \varnothing, \qquad \text{for } n_1 \neq n_2 \tag{24}$$

and

$$\text{b) } R_n(X) \subseteq X, \qquad \text{for } n = 0, 1, \cdots, N. \tag{25}$$

Proof: a) Equation (24) is a direct consequence of (4) and (8). b) Inequality (25) is a direct consequence of (5) and (8). □

Proposition 2 shows that the resulting morphological image representation subsets $R_n(X)$, for $n = 0, 1, \cdots, N$, are disjoint and antiextensive.

Proposition 3: If

$$B(n) \subseteq B(0), \qquad \text{for } n = 1, 2, \cdots, N - 1, \tag{26}$$

then

$$R(R_n(X)) = \{R_n(X)\}, \qquad \text{for } n = 0, 1, \cdots, N. \tag{27}$$

Proof: From (2) and (7) observe that, if $X \ominus B(0) = \varnothing$, we have $R_0(X) = X \ominus A(0) - \Psi_0[[X \ominus B(0)] \ominus A(0)] = X \ominus A(0) - \Psi_0[\varnothing] = X \ominus A(0)$, since $\Psi_0[\varnothing] = \varnothing$. Since also $A(0) = \{(0, 0)\}$ we obtain that $R_0(X) = X$. Observe from (2) that $A(n) = B(0) \oplus B(1) \oplus \cdots \oplus B(n - 1)$; therefore, we have that, if $X \ominus B(0) = \varnothing$, then $X \ominus A(n) = \varnothing$, for $n = 1, 2, \cdots, N$, which results to $R_n(X) = \varnothing$, for $n = 1, 2, \cdots, N$. Finally, from (9) we observe that, if $X \ominus B(0) = \varnothing$, then

$$R(X) = \{X\}. \tag{28}$$

Let us take a point $y \in R_n(X)$. From (7) we see that $y \in X \ominus A(n)$ and $y \notin \Psi_n[X \ominus A(n + 1)]$. From (6) we see that $y \notin X \ominus A(n + 1)$. Observe that $X \ominus A(n + 1) = [X \ominus A(n)] \ominus B(n)$; therefore, $y \notin [X \ominus A(n)] \ominus B(n)$. In this case, there exists a point $z \in B(n) \oplus \{y\}$ such that $z \notin X \ominus A(n)$. Since $z \in B(n) \oplus \{y\}$ and $B(n) \subseteq B(0)$, for $n = 0, 1, \cdots, N - 1$, we have that $z \in B(0) \oplus \{y\}$, from which we conclude that $B(0) \oplus \{y\} \not\subseteq R_n(X)$, for any $y \in R_n(X)$. In this case, we obtain that $R_n(X) \ominus B(0) = \varnothing$. Finally, by using (28) we prove (27). □

According to Proposition 3, when condition (26) is satisfied, a repeated application of the transformation $R(\bullet)$ does not influence the image representation. When (26) is not satisfied, the repeated application $R(\bullet)$ may result in a further reduction of the total cardinality of the representation. This is a desirable result in many applications of interest (e.g., image coding [22]). We illustrate this point in Fig. 1.

B. Morphological Image Representation and Morphological Openings

The set $\{X \bigcirc A(k), k = 0, 1, \cdots, N\}$ of the morphological openings of an image X by the sequence $\{A(n), n = 0, 1, \cdots, N\}$ of structuring elements provides a great deal of information about the shape and size of a given image. For example, the information derived from successive morphological openings may be used to develop useful size distributions (i.e., the granulometries in [15], [16]). It may also be utilized to provide a multiscale nonlinear smoothing of the boundary of an image [20], [23]. In this case, the image is represented in terms of the

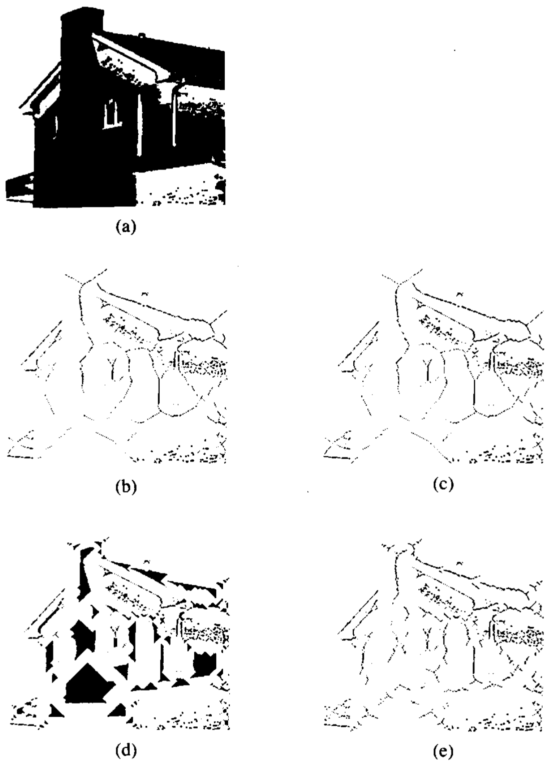

Fig. 1. (a) Original image; (b) the morphological image representation of (a) with $B(n) = B$, for $n = 0, 1, \cdots, N - 1$, and $\Psi_n[X] = (X \ominus nB) \bigcirc B$, for $n = 0, 1, \cdots, N$ (in this case B is the RHOMBUS structuring element in [17], and condition (26) is satisfied); (c) the morphological image representation of (b) with $B(n)$ and $\Psi_n[\bullet]$ as in (b), no difference is expected; (d) the morphological image representation of (a) with $B(n) = B \oplus B \oplus \cdots \oplus B$ (2^n times), for $n = 0, 1, \cdots, N - 1$, and $\Psi_n[X] = [X \ominus A(n)] \bigcirc B(n)$, for $n = 0, 1, \cdots, N$ (in this case B is as before and condition (26) is not satisfied); and, (e) the morphological image representation of (d) after 4 steps, with $B(n)$ with $\Psi_n[\bullet]$ as in (d).

discrete size transform, as it will be shown in the following subsection. We shall now show that there exists a strong relationship between $R(X)$ and the sequence $\{X \bigcirc A(k), k = 0, 1, \cdots, N\}$ of morphological openings.

Proposition 4:

a) In general,

$$X \bigcirc A(k) = \bigcup_{n=k}^{N} [R_n(X) \oplus A(n)], \qquad \text{for } k = 0, 1, \cdots, N. \tag{29}$$

b) In general,

$$R_n(X) = \varnothing, \qquad \text{for } n = 0, 1, \cdots, k - 1 \tag{30}$$

if and only if

$$X \bigcirc A(k) = X \tag{31}$$

and

$$\Psi_n[X \ominus A(n + 1)] = [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n), \qquad \text{for } n = 0, 1, \cdots, k - 1. \tag{32}$$

Proof: a) The proof of (29) is a direct consequence of (17) and (23). b) Assume that (30) is satisfied. Equation (31) is a direct consequence of (11), (29), and (30). Recall from (17) and (21) that $X \bigcirc A(n_2) \subseteq X \bigcirc A(n_1)$, for $n_1 \leq n_2$. Therefore,

$$X = X \bigcirc A(k) = X \bigcirc A(n + 1), \qquad \text{for } n = 0, 1, \cdots, k - 1. \tag{33}$$

From (6), (7), (30) and (33) we have

$$\Psi_n[X \ominus A(n + 1)] = X \ominus A(n) = [X \bigcirc A(n + 1)] \ominus A(n), \qquad \text{for } n = 0, 1, \cdots, k - 1. \tag{34}$$

Equation (32) is a direct consequence of (14) and (34).

Assume now that (31) and (32) are satisfied. From (14) and (33) we have

$$\begin{aligned}\Psi_n[X \ominus A(n + 1)] &= [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \\ &= [X \bigcirc A(n + 1)] \ominus A(n) \\ &= X \ominus A(n)\end{aligned} \tag{35}$$

for $n = 0, 1, \cdots, k - 1$. Condition (30) is a direct consequence of (7) and (35). □

From Part a) we see that if we omit the k first morphological image representation subsets $R_n(X)$, for $n = 0, 1, \cdots, k - 1$, from the inverse formula (11), we do not reconstruct the image X but a smoothed version of X which is the opening of image X by the structuring element $A(k)$. From Part b) we see that the k first morphological image representation subsets $R_n(X)$, for $n = 0, 1, \cdots, k - 1$, are empty, if and only if the mapping $\Psi_n[\bullet]$ is constrained to the values given by (32), and the image X is $A(k)$-open (see (31)).

C. Morphological Image Representation and Pattern Spectrum

Another important image transformation is the discrete size transform given by $D_n(X) = X \bigcirc A(n) - X \bigcirc A(n + 1)$, for $n = 0, 1, \cdots, N$. A special form of this transform is introduced in [19] as a shape-size image descriptor. Observe that: a) the elements of the sequence $\{D_n(X), n = 0, 1, \cdots, N\}$ are disjoint; i.e., $D_{n1}(X) \cap D_{n2}(X) = \varnothing$, for $n_1 \neq n_2$; and, b) $X = \cup_{n=0}^{N} D_n(X)$. Therefore, $\{D_n(X), n = 0, 1, \cdots, N\}$ decomposes the image X into $N + 1$ disjoint subsets. The importance of this transform stems from the fact that the discrete pattern spectrum (a very useful quantity for shape analysis) is given by

$$\begin{aligned}[PS(X)](n, B(n)) &= \text{Card}\,[D_n(X)] \\ &= \text{Card}\,[X \bigcirc A(n) - X \bigcirc A(n + 1)]\end{aligned} \tag{36}$$

for $n = 0, 1, \cdots, N$. A special form of the discrete pattern spectrum is introduced in [19], [20]. The discrete pattern spectrum conveys useful information about the shape-size content of an image. From (36) it is clear that $[PS(X)](n, B(n)) = 0$ if and only if $D_n(X) = \varnothing$; that is, when the nth component $D_n(X)$ of the discrete size transform is empty, the nth component of the discrete pattern spectrum is equal to zero, and vice versa.

In the next proposition we shall establish the relationship between $R(X)$ and the discrete pattern spectrum, thereby increasing the utility of $R(X)$ as a shape-pattern analysis tool.

Proposition 5: In general,

$$R_n(X) = \varnothing \tag{37}$$

if and only if

$$\Psi_n[X \ominus A(n+1)] = [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \tag{38}$$

and

$$[PS(X)](n, B(n)) = 0. \tag{39}$$

Proof: Assume that (37) is satisfied. From (6) and (7) we have

$$X \ominus A(n) = \Psi_n[X \ominus A(n+1)]. \tag{40}$$

Condition (38) is a direct result of (10), (40), and the fact that $[[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \subseteq X \ominus A(n)$. By using (2b), (38), and (40) we can easily prove that

$$\begin{aligned} X \bigcirc A(n) &= [[[X \ominus A(n)] \bigcirc B(n)] \bullet A(n)] \oplus A(n) \\ &= [[X \ominus A(n+1)] \oplus A(n+1)] \bigcirc A(n) \\ &\subseteq X \bigcirc A(n+1). \end{aligned} \tag{41}$$

From (36) and (41) we prove (39).

Assume now that (38) and (39) are satisfied. In this case, observe that $X \bigcirc A(n) = X \bigcirc A(n+1)$. By using (13) and (14) we have

$$\begin{aligned} X \ominus A(n) &= [X \bigcirc A(n)] \ominus A(n) \\ &= [X \bigcirc A(n+1)] \ominus A(n) \\ &= [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n). \end{aligned} \tag{42}$$

From (7), (38), and (42) we obtain (37). □

From Proposition 5 we see that, the nth morphological image representation subset $R_n(X)$ is empty, if and only if the mapping $\Psi_n[\bullet]$ is constrained by (38), and the nth component of the pattern-spectrum is equal to zero (see (39)).

From the previous analysis we see that any choice for the sequence of structuring elements $\{B(n), n = 0, 1, \cdots, N-1\}$ and any choice for the sequence of set transformations $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$ that satisfy (10) will result in an invertible morphological representation $R(X)$. In this case $R(X)$ decomposes an image X into a sequence of $N+1$ disjoint subsets $\{R_n(X), n = 0, 1, \cdots, N\}$, which uniquely characterize X. Different choices for $B(n)$ and $\Psi_n[\bullet]$ will result in different image representations. A discussion on the effects of choosing different sequences of structuring elements $\{B(n), n = 0, 1, \cdots, N-1\}$ appears in [18], [22], and [24]. We shall now study the effects of choosing different sequences of transformations $\{\Psi_n[\bullet], n = 0, 1, \cdots, N\}$.

D. Morphological Image Representation Examples

The following examples are important special cases of the general morphological image representation $R(X)$.

Example 1 (Generalized Morphological Skeleton [18]): Consider

$$\Psi_n[X] = X \oplus B(n), \qquad \text{for } n = 0, 1, \cdots, N. \tag{43}$$

From (2b) and (43) we have

$$\begin{aligned} \Psi_n[X \ominus A(n+1)] &= [X \ominus A(n+1)] \oplus B(n) \\ &= [X \ominus A(n)] \bigcirc B(n) \\ &\subseteq [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \end{aligned}$$

and

$$\Psi_n[X \ominus A(n+1)] = [X \ominus A(n+1)] \oplus B(n) \supseteq X \ominus A(n+1).$$

Therefore, the mapping $\Psi_n[\bullet]$ satisfies (10). In this case,

$$R_s(X) = \{R_{s,0}(X), R_{s,1}(X), \cdots, R_{s,N}(X)\} \tag{44a}$$

where

$$R_{s,n}(X) = X \ominus A(n) - [X \ominus A(n)] \bigcirc B(n). \tag{44b}$$

From (11) we obtain

$$X = \bigcup_{n=0}^{N} [R_{s,n}(X) \oplus A(n)].$$

Example 2: Let us now consider

$$\Psi_n[X] = X \oplus [B(n) \bullet A(n)], \qquad \text{for } n = 0, 1, \cdots, N. \tag{45}$$

Observe that

$$\begin{aligned} \Psi_n&[X \ominus A(n+1)] \\ &= [X \ominus A(n+1)] \oplus [B(n) \bullet A(n)] \\ &= [X \ominus A(n+1)] \oplus [[B(n) \oplus A(n)] \ominus A(n)] \\ &\subseteq [[X \ominus A(n+1)] \oplus [B(n) \oplus A(n)]] \ominus A(n) \\ &= [[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) \end{aligned}$$

and

$$\begin{aligned} \Psi_n&[X \ominus A(n+1)] \\ &= [X \ominus A(n+1)] \oplus [B(n) \bullet A(n)] \\ &\supseteq [X \ominus A(n+1)] \oplus B(n) \\ &\supseteq X \ominus A(n+1) \end{aligned}$$

where we have used (2b) and (45). It is now clear that (45) satisfies (10). In this case,

$$R_c(X) = \{R_{c,0}(X), R_{c,1}(X), \cdots, R_{c,N}(X)\} \tag{46a}$$

where

$$R_{c,n}(X) = X \ominus A(n) - [[X \ominus A(n+1)] \oplus [B(n) \bullet A(n)]]. \quad (46b)$$

From (11) we also obtain

$$X = \bigcup_{n=0}^{N} [R_{c,n}(X) \oplus A(n)].$$

From (43)–(46) observe that $R_{c,n}(X) \subseteq R_{s,n}(X)$, for $n = 0, 1, \cdots, N$.

Example 3: (Generalized reduced morphological skeleton [19], [20]): Finally, let us consider

$$\Psi_n[X] = [X \oplus B(n)] \bullet A(n), \qquad \text{for } n = 0, 1, \cdots, N. \quad (47)$$

From (2b) and (47) we have

$$\Psi_n[X \ominus A(n+1)] = [[X \ominus A(n+1)] \oplus B(n)] \bullet A(n) = [[X \ominus A(n)] \circ B(n)] \bullet A(n)$$

and

$$\Psi_n[X \ominus A(n+1)] = [[X \ominus A(n+1)] \oplus B(n)] \bullet A(n) \supseteq [X \ominus A(n+1)] \oplus B(n) \supseteq X \ominus A(n+1).$$

Therefore (47) satisfies (10). In this case,

$$R_{\min}(X) = \{R_{\min,0}(X), R_{\min,1}(X), \cdots, R_{\min,N}(X)\} \quad (48a)$$

where

$$R_{\min,n}(X) = X \ominus A(n) - [[X \ominus A(n)] \circ B(n)] \bullet A(n). \quad (48b)$$

From (11) we obtain

$$X = \bigcup_{n=0}^{N} [R_{\min,n}(X) \oplus A(n)].$$

Observe that $R_{\min,n}(X) \subseteq R_n(X)$, for every mapping $\Psi_n[\bullet]$ that satisfies (10). In this case,

$$\text{Card } [R_{\min,n}(X)] \leq \text{Card } [R_n(X)], \qquad \text{for } n = 0, 1, \cdots, N$$

and for every image $X \subseteq Z^2$.

III. Morphological Skeleton Representation of Discrete and Binary Images

Among all possible morphological transforms, a very useful choice for image representation is the morphological skeleton [7]–[9], [16]–[20]. The morphological skeleton of a discrete, binary image $X \subseteq Z^2$ contains sufficient information about X (i.e., the image X can be exactly reconstructed from its morphological skeleton) and it has been effectively used in many areas of image analysis.

Various definitions of morphological skeletons of discrete and binary images have appeared in the literature [16]–[20]. The purpose of this section is to study the relationship between the morphological image representation $R(X)$ and the various forms of morphological skeletons. As a result of this study, we develop a unified theory for the mathematical description of the morphological skeleton decomposition of discrete and binary images.

A. Discrete Medial Axis Transform

The skeleton (or, medial axis) of a continuous, binary image X is defined as the locus of the centers of the maximal disks which can be inscribed in X [2], [16]. We say that $D \oplus \{x\}$ is a maximal disk in X if we cannot find a larger disk (not necessarily centered at x) which is included in X and contains $D \oplus \{x\}$. The locus of the centers of the maximal disks D_r, of radius r, that can be inscribed in X is known as the skeleton subset S_r of order r. It is clear that the skeleton is the union of all skeleton subsets of order r, for $r > 0$. One of the most important properties of the skeleton of a continuous, binary image is the fact that the original image X can be exactly reconstructed from its skeleton subsets. Indeed, it can be easily shown that

$$X = \bigcup_{r>0} \bigcup_{z \in S_r} D_r \oplus \{z\}.$$

Another interesting property is the fact that the skeleton can be easily expressed in terms of a sequence of successive morphological erosions and openings [16].

In order to develop a similar tool in the digital domain, we present the following definitions.

Definition 2: Consider a sequence $\{B(n), n = 0, 1, \cdots, N-1\}$ of structuring elements containing the origin. The structuring element $A(n) \oplus \{z\}$, with $A(n)$ given by (2), is a maximal structuring element in a discrete, binary image $X \subseteq Z^2$, if $A(n) \oplus \{z\} \subseteq X$ and there exists no other structuring element $A(m) \oplus \{y\}$, with $m > n$, such that $A(n) \oplus \{z\} \subseteq A(m) \oplus \{y\} \subseteq X$.

Definition 3: Consider a sequence $\{B(n), n = 0, 1, \cdots, N-1\}$ of structuring elements containing the origin. The discrete medial axis transform $M(X)$ of a discrete, binary image $X \subseteq Z^2$ is defined by

$$M(X) = \{M_0(X), M_1(X), \cdots, M_N(X)\} \quad (49)$$

where $M_n(X)$ is the discrete medial axis subset of order n given by

$$M_n(X) = \{z: A(n) \oplus \{z\} \in \Omega(X)\}. \quad (50)$$

In (50) $A(n)$ is a structuring element given by (2), whereas $\Omega(X)$ is the collection of all maximal structuring elements in X. From the previous definitions we have

$$M_n(X) \subseteq M(X) \subseteq X, \qquad \text{for } n = 0, 1, \cdots, N \quad (51a)$$

and

$$M_{n1}(X) \cap M_{n2}(X) = \varnothing, \qquad \text{for } n_1 \neq n_2. \quad (51b)$$

Therefore, the discrete medial axis transform of an image X is always a subset of X, whereas, the various discrete medial axis subsets are disjoint.

The definition of $M(X)$, given by (49) and (50), closely resembles the definition of the skeleton in the continuous domain. In the discrete domain, the structuring element $A(n)$ of size n is analogous to the maximal disk D_r of radius r. From (51) observe that the discrete medial axis transform represents the underlying image by a collection of $N + 1$ disjoint subsets $\{M_n(X), n = 0, 1, \cdots, N\}$ whose cardinality is always less than the cardinality of image X. We shall now study the relationship between the morphological image representation $R(X)$ and the medial axis transform.

B. Morphological Image Representation and Discrete Medial Axis Transform

Since $R(X)$ is an image representation that guarantees exact reconstruction of the image under consideration, and since it also satisfies properties similar to (51), it would be a reasonable choice for the mathematical description of the skeleton of a discrete, binary image. The only additional requirement for the representation $R(X)$ is to become equivalent to the discrete medial axis transform. We shall examine this possibility next.

Proposition 6:

a) If

$$\Psi_n[X \ominus A(n+1)] \subseteq [X \ominus A(n+1)] \oplus B(n) \quad (52)$$

then

$$M_n(X) \subseteq R_n(X). \quad (53)$$

b) If

$$M_n(X) \supseteq R_n(X) \quad (54)$$

then

$$\Psi_n[X \ominus A(n+1)] \supseteq [X \ominus A(n+1)] \oplus B(n). \quad (55)$$

Proof: a) Assume that $z \in M_n(X)$ and $z \notin R_n(X)$. From (50) it is clear that $A(n) \oplus \{z\} \subseteq X$; therefore, $z \in X \ominus A(n)$. Since $z \notin R_n(X)$, we see from (7) that $z \in \Psi_n[X \ominus A(n+1)]$, which together with (52) results in $z \in [X \ominus A(n)] \bigcirc B(n)$. Therefore, there exists a point y such that

$$z \in B(n) \oplus \{y\} \subseteq X \ominus A(n). \quad (56)$$

From (2b) we have that $A(n+1) \oplus \{y\} = [B(n) \oplus \{y\}] \oplus A(n)$ which, together with (56) gives

$$A(n+1) \oplus \{y\} \subseteq [X \ominus A(n)] \oplus A(n) \subseteq X. \quad (57)$$

From (56) we also obtain that $A(n) \oplus \{z\} \subseteq A(n) \oplus [B(n) \oplus \{y\}]$ which, together with (2b) and (57) implies that

$$A(n) \oplus \{z\} \subseteq A(n+1) \oplus \{y\} \subseteq X. \quad (58)$$

Observe that (58) contradicts the assumption that $A(n) \oplus \{z\} \in \Omega(X)$. Therefore, $z \notin \Psi_n[X \ominus A(n+1)]$, and since $z \in X \ominus A(n)$, we obtain from (7) that $z \in R_n(X)$. This proves (53).

b) Let us now take a point $z \in [X \ominus A(n+1)] \oplus B(n) = [X \ominus A(n)] \bigcirc B(n)$. We now see that $z \in X \ominus A(n)$. Assume that $z \notin \Psi_n[X \ominus A(n+1)]$. From (7) we have that $z \in R_n(X)$, which, together with (54), implies that $z \in M_n(X)$. Therefore, $A(n) \oplus \{z\} \in \Omega(X)$; i.e., $A(n) \oplus \{z\}$ is a maximal structuring element in X. Since $z \in [X \ominus A(n)] \bigcirc B(n)$, there exists a point y such that (56) is satisfied, which in turn results in (58). Equation (58) contradicts the fact that $A(n) \oplus \{z\}$ is a maximal structuring element. Therefore, $z \in \Psi_n[X \ominus A(n+1)]$, which proves (55). □

As we see from Proposition 6, when $\Psi_n[X] \subseteq X \oplus B(n)$, then $R_n(X)$ contains $M_n(X)$. When $\Psi_n[X] \supseteq X \oplus B(n)$, then $M_n(X)$ may not be included in $R_n(X)$. A stronger relationship will be established in the following. The next lemma is important. Its proof can be found in the Appendix.

Lemma 1: If $Y = X \oplus B$ and X is a convex[3] and bounded set, then

$$X = Y \ominus B. \quad (59)$$

Proposition 7: If i) $B(n)$ is convex and bounded, for $n = 0, 1, \cdots, N-1$; and ii) $B(n) \oplus B(n+1) \oplus \cdots \oplus B(m)$ is convex and bounded, for all n and $m > n$, then, for every $n = 0, 1, \cdots, N$: a) a necessary and sufficient condition for (53) to be satisfied is (52); b) a necessary and sufficient condition for (54) to be satisfied is (55); and c) a necessary and sufficient condition for $M_n(X) = R_n(X)$ is given by $\Psi_n[X \ominus A(n+1)] = [X \ominus A(n+1)] \oplus B(n)$.

Proof: a) The sufficiency of condition (52) is a direct consequence of Proposition 6a. Assume now that (53) is satisfied. Let us take a point $z \in \Psi_n[X \ominus A(n+1)]$. From (7) we know that $z \notin R_n(X)$, which, together with (53), implies that $z \notin M_n(X)$. Therefore, $A(n) \oplus \{z\} \notin \Omega(X)$; i.e., $A(n) \oplus \{z\}$ is not a maximal structuring element in X. In this case there exists a point y and an integer $m > n$, such that (see also Definition 2)

$$A(n) \oplus \{z\} \subseteq A(m) \oplus \{y\} \subseteq X, \quad \text{for } m > n. \quad (60)$$

It can be easily verified that $\{z\} = [A(n) \oplus \{z\}] \ominus A(n)$, which, together with (60) gives $\{z\} \subseteq [A(m) \oplus \{y\}] \ominus A(n) = [A(m) \ominus A(n)] \oplus \{y\}$, for $m > n$. Let $D(m, n) = A(m) \ominus A(n)$, *for* $m > n$. In this case,

$$\{z\} \subseteq D(m, n) \oplus \{y\}, \quad \text{for } m > n. \quad (61)$$

Observe that $D(m, n) \oplus \{y\} = [A(m) \oplus \{y\}] \ominus A(n)$, which, together with (60), gives $D(m, n) \oplus \{y\} \subseteq X \ominus A(n)$, for $m > n$. This last relationship implies that

$$D(m, n) \oplus \{y\} \subseteq [X \ominus A(n)] \bigcirc D(m, n), \quad \text{for } m > n. \quad (62)$$

[3]A set B is a convex set when it is identical to the intersection of all half planes which contain B (i.e., its convex hull). For more information about convex sets see [16].

From (61) and (62) we obtain

$$\{z\} \subseteq [X \ominus A(n)] \bigcirc D(m, n), \qquad \text{for } m > n. \quad (63)$$

Now let $E(m, n) = B(m - 1) \oplus B(m - 2) \oplus \cdots \oplus B(n)$, for $m > n$. Observe that, $A(m) = E(m, n) \oplus A(n)$. Since $E(m, n)$ is bounded and convex by assumption ii), we have from (59) that $E(m, n) = A(m) \ominus A(n) = D(m, n)$. Observe that in this case $E(m, n)$ is $B(n)$-open; therefore, $D(m, n)$ is $B(n)$-open and we have that

$$[X \ominus A(n)] \bigcirc D(m, n) \subseteq [X \ominus A(n)] \bigcirc B(n). \quad (64)$$

Finally, from (63) and (64) we obtain $\{z\} \subseteq [X \ominus A(n)] \bigcirc B(n)$, which proves the necessity of condition (52).

b) The necessity of condition (55) is a direct consequence of Proposition 6b). Assume now that condition (55) is satisfied. Take a point $z \in R_n(X)$. In this case $z \notin \Psi_n[X \ominus A(n + 1)]$, which, together with (55), implies that $z \notin [X \ominus A(n)] \bigcirc B(n)$. Now assume that $z \notin M_n(X)$. In this case, $A(n) \oplus \{z\}$ is not a maximal structuring element in X. From the proof of part a) we have that $z \in [X \ominus A(n)] \bigcirc B(n)$, which is a contradiction. This proves the sufficiency of condition (55).

c) The proof is a direct consequence of Parts a and b. □

Condition ii) of Proposition 7 is necessary, since the dilation of two convex sets is not always convex [16]. The next corollary is a direct consequence of Proposition 7.

Corollary 1: If the structuring element $B(n)$ satisfies conditions i) and ii) of Proposition 7, then:

$$M(X) = R(X) \qquad \text{if and only if } \Psi_n[X \ominus A(n + 1)] = [X \ominus A(n + 1)] \oplus B(n). \quad (65)$$

It is clear from (65) that, when conditions i) and ii) of Proposition 7 are satisfied, the morphological image representation $R(X)$ becomes equivalent to the discrete medial axis transform only when $\Psi_n[X] = X \oplus B(n)$, for $n = 0, 1, \cdots, N$. In this case, $R(X)$ can be used to mathematically describe a digital version of the skeleton of binary images. This skeleton has been proposed in [18] as the generalized discrete morphological skeleton transform, and it satisfies all the properties mentioned in Example 1. A special case of this transform is obtained when $B(n) = B$, for $n = 0, 1, \cdots, N - 1$. In this case $A(n) = nB = B \oplus B \oplus \cdots \oplus B$ (n times), and, therefore, $\Psi_n[X \ominus A(n + 1)] = (X \ominus nB) \bigcirc B$, for $n = 0, 1, \cdots, N$. The resulting morphological skeleton has been used in [17]–[20].

When a reduced cardinality skeleton is of interest, the morphological image representation $R(X)$ can provide such a skeleton under some constraints. As we demonstrated in Example 3, the choice (47) leads to the representation $R_{\min}(X)$. From Example 3 we see that $R_{\min}(X)$ provides the smallest cardinality from all possible morphological image representations $R(X)$. In [19], [20] a special case of $R_{\min}(X)$ is referred to as the reduced morphological skeleton transform. From Proposition 6, and since $[X \oplus B(n)] \bullet A(n) \supseteq X \oplus B(n)$, it is clear that the discrete medial axis transform may contain points that do not belong to $R_{\min}(X)$. Therefore, $R_{\min}(X)$ may not qualify for the mathematical description of the discrete morphological skeleton. Before we discuss a solution to this problem we shall introduce a new definition.

Definition 4: If the set B is C-open, then the opening $X \bigcirc B$ is a C-connected opening if for every $C \oplus \{x\} \subseteq X \bigcirc B$, $x \in Z^2$, there exists a $y \in Z^2$ such that $C \oplus \{x\} \subseteq B \oplus \{y\} \subseteq X \bigcirc B$.

The following lemma will be useful for the proof of Proposition 8. Its proof can be found in the Appendix.

Lemma 2: If the set B is C-open, then $X \bigcirc B$ is a C-connected opening if and only if

$$(X \bigcirc B) \ominus C = (X \ominus B) \oplus (B \ominus C). \quad (66)$$

We have the following proposition.

Proposition 8: If $B(n)$ is a convex and bounded structuring element and if the opening $X \bigcirc A(n + 1)$ is $A(n)$-connected, then

$$R_{\min,n}(X) = R_{s,n}(X). \quad (67)$$

Proof: From (2b), the convexity and boundness of $B(n)$ and Lemma 1 we obtain

$$B(n) = A(n + 1) \ominus A(n). \quad (68)$$

From (2b), (66), and (68) we have

$$\begin{aligned}
[[X \ominus A(n)] \bigcirc B(n)] \bullet A(n) &= [[X \ominus A(n + 1)] \oplus A(n + 1)] \ominus A(n) \\
&= [X \bigcirc A(n + 1)] \ominus A(n) \\
&= [X \ominus A(n + 1)] \oplus [A(n + 1) \ominus A(n)] \\
&= [X \ominus A(n)] \bigcirc B(n)
\end{aligned}$$

which, together with (44) and (48) proves (67). □

The following corollary establishes the equivalence between $R_{\min}(X)$ and the discrete morphological skeleton transform. Its proof is a direct consequence of Corollary 1 and Proposition 8.

Corollary 2: If the structuring element $B(n)$ satisfies conditions i) and ii) of Proposition 7 and if the opening $X \bigcirc A(n + 1)$ is $A(n)$-connected, for $n = 0, 1, \cdots, N - 1$, then $R_{\min}(X) = M(X)$.

According to Corollary 2, and under the appropriate conditions, $R_{\min}(X)$ can be used to mathematically describe the discrete medial axis transform while it retains all the important properties of the morphological image representation $R(X)$.

IV. Summary

In this paper we presented a unified theory for the morphological representation of discrete and binary images, which allowed us to study the relationship between the various forms of morphological skeletons. The most important aspect of our work was the definition of the morphological image representation $R(X)$, a general set transform for the representation of discrete and binary images. We presented a systematic study of its properties and we established its relationship with some important shape analysis tools. It was also shown that many useful forms of morphological skeletons can be derived as special cases of our representation.

Appendix
Proofs of Lemmas

Proof of Lemma 1: Observe that

$$Y \ominus B = (X \oplus B) \ominus B = X \bullet B. \tag{A1}$$

Since X is a convex and bounded set, then X is B-closed (see [15, Proposition 1-5-3]). Therefore,

$$X \bullet B = X. \tag{A2}$$

By using (A1) and (A2) we obtain (59). □

Proof of Lemma 2: Let us first assume that (66) is satisfied. Let us also take a point $x \in Z^2$ such that $C \oplus \{x\} \subseteq X \bigcirc B$. It is easy to see that $\{x\} \subseteq (X \bigcirc B) \ominus C$, which, together with (66) gives

$$x \in (X \ominus B) \oplus (B \ominus C). \tag{A3}$$

Observe now that

$$X \bigcirc B = \bigcup_{y \in X \ominus B} B \oplus \{y\}. \tag{A4}$$

Because B is assumed to be C-open we also have

$$B = B \bigcirc C = \bigcup_{z \in B \ominus C} C \oplus \{z\}. \tag{A5}$$

Since (A3) is true, there exists a point $y^* \in X \ominus B$ and a point $z^* \in B \ominus C$ such that $x = y^* + z^*$. Therefore, from (A4) and (A5) we obtain

$$\begin{aligned} C \oplus \{x\} &= [C \oplus \{z^*\}] \oplus \{y^*\} \\ &\subseteq \left[\bigcup_{z \in B \ominus C} C \oplus \{z\} \right] \oplus \{y^*\} \\ &= B \oplus \{y^*\} \subseteq \bigcup_{y \in X \ominus B} B \oplus \{y\} = X \bigcirc B, \end{aligned}$$

or

$$C \oplus \{x\} \subseteq B \oplus \{y^*\} \subseteq X \bigcirc B. \tag{A6}$$

Definition 4 and (A6) prove the fact that the opening $X \bigcirc B$ is C-connected.

Let us now assume that the opening $X \bigcirc B$ is C-connected. We have

$$\begin{aligned} (X \bigcirc B) \ominus C &= [(X \ominus B) \oplus B] \ominus C \\ &\supseteq (X \ominus B) \oplus (B \ominus C). \end{aligned} \tag{A7}$$

Let us also assume that $x \in (X \bigcirc B) \ominus C$. In this case $C \oplus \{x\} \subseteq X \bigcirc B$. Since the opening $X \bigcirc B$ is C-connected, we have that there exists a point $y \in Z^2$ such that

$$C \oplus \{x\} \subseteq B \oplus \{y\} \subseteq X \bigcirc B. \tag{A8}$$

From (A8) we have that

$$\begin{aligned} \{x\} &= (C \ominus C) \oplus \{x\} = (C \oplus [x]) \ominus C \\ &\subseteq (B \oplus \{y\}) \ominus C = (B \ominus C) \oplus \{y\} \end{aligned}$$

or

$$x = b + y, \quad \text{for } b \in B \ominus C. \tag{A9}$$

From (A8) we also have that

$$\begin{aligned} \{y\} &= (B \ominus B) \oplus \{y\} = (B \oplus \{y\}) \ominus B \\ &\subseteq (X \bigcirc B) \ominus B \subseteq X \ominus B. \end{aligned} \tag{A10}$$

From (A9) and (A10) we finally obtain

$$x = b + y, \quad \text{for } b \in B \ominus C \quad \text{and } y \in X \ominus B$$

or

$$x \in (X \ominus B) \oplus (B \ominus C).$$

Therefore,

$$(X \bigcirc B) \ominus C \subseteq (X \ominus B) \oplus (B \ominus C). \tag{A11}$$

Inequalities (A7) and (A11) prove (66). □

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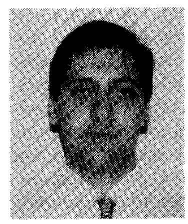

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