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On the Convergence and Roots of Order-Statistics Filters

Mohammed Charif-Chefchaouni and Dan Schonfeld

***Abstract*—In this correspondence, we propose a comprehensive theory of the convergence and characterization of roots of order-statistics filters. Conditions for the convergence of iterations of order-statistics filters are proposed. Criteria for the morphological characterization of roots of order-statistics filters are also proposed.**

I. Introduction

Order-statistics filters (and their repeated iterations) have been demonstrated to be very useful in signal and image processing [1]–[3]. The utility of these filters has been motivated by their theoretical properties, e.g., the median filter minimizes the absolute-value of the error [1]. The ubiquity of order-statistics filters (and their repeated iterations) is further attributed to their superior performance in numerous signal and image processing applications, e.g., restoration of noisy images [2]–[3].

Despite the merit of order-statistics filters (and their repeated iterations) in numerous signal and image processing applications, their practical utility is often limited due to the indefiniteness of their iterations, i.e., in general, iterations of order-statistics filters do not converge [4]–[7]. This limitation is often manifested in the form of oscillations of iterations of order-statistics filters. As a consequence of their inherent oscillations, the computational efficiency of the implementation of iterations of order-statistics filters is very low. An increase in the computational efficiency of the implementation of iterations of order-statistics filters could be attained by exploiting conditions for their convergence. Although some conditions for the convergence of iterations of order-statistics filters have been proposed, in general, this topic has remained an open problem [4]–[7].

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A *root* of an order-statistics filter is used to designate its fixed point [4]–[10]. An important example of a root of an order-statistics filter is obtained by the convergence of its iterations. A characterization of roots of order-statistics filters consequently has some important ramifications on the convergence of their iterations. These ramifications are particularly crucial to the representation of the class of signals (images) obtained by the convergence of iterations of order-statistics filters. Although some criteria for the characterization of roots of order-statistics filters have been proposed, in general, this topic has remained an open problem as well [4]–[10].

In this correspondence, a comprehensive theory of the convergence and characterization of roots of order-statistics filters is proposed. In Section II, we provide some elementary definitions used throughout this presentation. In Section III, we introduce morphological bounds on order-statistics filters. Conditions for openings and closings to serve as bounds (lower and upper, respectively) on order-statistics filters are presented. These bounds are essential to the derivation of conditions for the convergence of iterations of order-statistics filters and the characterization of their roots provided in the subsequent sections of this presentation. In Section IV, we investigate the convergence of iterations of order-statistics filters. Conditions for the convergence of iterations of order-statistics filters are proposed. In Section V, we investigate the roots of order-statistics filters. Criteria for the morphological characterization of roots of order-statistics filters are proposed. Finally, in Section VI, we present a brief summary and discussion of our results.

II. Preliminaries

In this presentation, we shall restrict ourselves to the collection of n-dimensional discrete and binary sets, i.e., $\{X : X \subseteq \mathcal{Z}^n\}$. The *translation* X_y of a set X by vector y is defined by $X_y = \{z : z = x + y, x \in X\}$. The *complement* X^c of a set X is defined by $X^c = \{x \in \mathcal{Z}^n : x \notin X\}$. The *symmetric* X^s of a set X is defined by $X^s = \{-x : x \in X\}$. The *cardinality* $|X|$ of a set X will denote the total number of elements contained in the set. A set X is *finite* if $|X| < \infty$.

Let the *structuring element* $B \subseteq \mathcal{Z}^n$ denote a finite n-dimensional discrete and binary set. Let us use $\gamma_B(X)$ and $\phi_B(X)$ to denote the *opening* and *closing* of a set X by a structuring element B, respectively, [11]–[13]. A set X is *B-open* (resp., *B-closed*) if $\gamma_B(X) = X$ (resp., $\phi_B(X) = X$).

Let the *window* $B \subseteq \mathcal{Z}^n$ denote a finite n-dimensional discrete and binary set. The mth *order statistics* (OS) (resp., *median*) filter mth $OS(X; B)$ (resp., $med(X; B)$) of a set X with respect to window B is given by

$$m\text{th}\, OS(X; B) = \{y \in \mathcal{Z}^n : |X \cap B_y| \geq m\}$$

for $m = 1, \cdots, |B|$ (resp., $m = (|B| + 1)/2$ when $|B|$ is odd).

III. Bounds

In this section, we introduce morphological bounds on order-statistics filters. Conditions for openings and closings to serve as bounds (lower and upper, respectively) on order-statistics filters are

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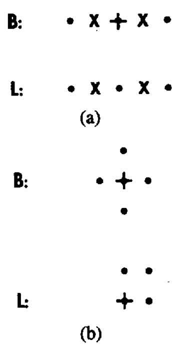

Fig. 1. Structuring elements: (a) $L \subseteq_2 B$ ($\bullet \in X$ and $\times \in X^c$); (b) $L \subseteq_3 B$ (sets B and L are the "rhombus" and "boxne," respectively).

presented. These bounds are essential to the derivation of conditions for the convergence of iterations of order-statistics filters and the characterization of their roots provided in the subsequent sections of this presentation. A thorough investigation of morphological bounds on order-statistics filters appears in [14].

Let the structuring element $L \subseteq \mathcal{Z}^n$ and window $B \subseteq \mathcal{Z}^n$ denote finite n-dimensional discrete and binary sets.

Definition 1: A set L is *m-included* in set B ($L \subseteq_m B$) if

$$|L \cap B_y| \geq m,$$

for every $y \in L$.

Lemma 1 [14]: $L \subseteq_m B$ if and only if $L \subseteq m^{th}OS(L; B)$.

Example 1(a): Consider sets B and L depicted in Fig. 1(a). From Definition 1 (or Lemma 1), we observe that $L \subseteq_2 B$.

Example 1(b): Consider sets B and L depicted in Fig. 1(b) (i.e., the "rhombus" and "boxne," respectively). From Definition 1 (or Lemma 1), we observe that $L \subseteq_3 B$.

In the following, we present conditions for openings and closings to serve as bounds (lower and upper, respectively) on order-statistics filters.

Property 1 [14]: There exist sets L and B such that $L \subseteq_m B$ if and only if

$$\gamma_L(X) \subseteq m^{th}OS(X; B)$$

for every $X \subseteq \mathcal{Z}^n$.

Property 2 [14]: There exist sets L and B such that $L \subseteq_{(|B|-m+1)} B$ if and only if

$$m^{th}OS(X; B) \subseteq \phi_L(X)$$

for every $X \subseteq \mathcal{Z}^n$.

Property 3 [14]: There exist sets L and B such that $L \subseteq_m B$ (resp., $L \subseteq_{(|B|-m+1)} B$) whenever $m \geq (|B| + 1)/2$ (resp., $m \leq (|B| + 1)/2$) if and only if

$$\gamma_L(X) \subseteq m^{th}OS(X; B) \subseteq \phi_L(X),$$

for every $X \subseteq \mathcal{Z}^n$.

Example 2: Let us consider a set $X \subseteq \mathcal{Z}$. Consider convex and symmetric set B such that $|B| = 2M + 1$. Consider also convex set L such that $|L| = M + 1$. From Definition 1 (or Lemma 1), we observe that $L \subseteq_{M+1} B$. From Property 3 (and the fact that $M + 1 = (|B| + 1)/2$ where $|B| = 2M + 1$), we observe that $\gamma_L(X) \subseteq med(X; B) \subseteq \phi_L(X)$. The morphological (lower and upper) bounds $\gamma_L(X)$ and $\phi_L(X)$ (respectively) on the median filter $med(X; B)$ are depicted in Fig. 2.

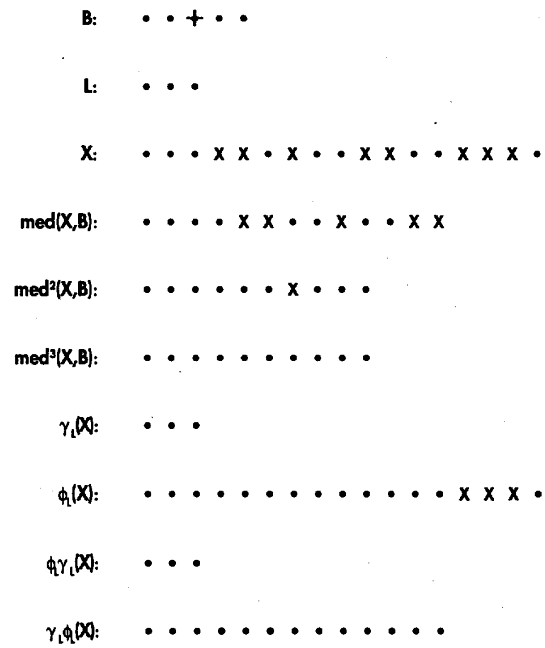

Fig. 2. Morphological representation of median filters (1-D signals): Morphological (lower and upper) bounds $\gamma_L(X)$ and $\phi_L(X)$ (respectively) on the median filter $med(X; B)$. Morphological (lower and upper) bounds $\phi_L\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ (respectively) on the iteration of the median filter $med^n(X; B)$. Convergence of iterations of the median filter $\{med^n(X; B) : n \in \mathcal{N}\}$. The root $med^3(X; B)$ of the median filter with respect to window B. The roots $\phi_L(\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ of the median filter with respect to window B. ($\bullet \in X$ and $\times \in X^c$.)

Example 3: Let us consider a set $X \subseteq \mathcal{Z}^2$. Consider sets B and L depicted in Fig. 1(b) (i.e., the "rhombus" and "boxne," respectively). From Example 1(b), we observe that $L \subseteq_3 B$. From Property 3 (and the fact that $3 = (|B| + 1)/2$, where $|B| = 5$), we observe that $\gamma_L(X) \subseteq med(X; B) \subseteq \phi_L(X)$. The morphological (lower and upper) bounds $\gamma_L(X)$ and $\phi_L(X)$ (respectively) on the median filter $med(X; B)$ are depicted in Fig. 3.

The morphological bounds on order-statistics filters proposed above will be used in the derivation of conditions for the convergence of iterations of order-statistics filters and the characterization of their roots provided in the subsequent sections of this presentation.

IV. Convergence

In this section, we investigate the convergence of iterations of order-statistics filters. Conditions for the convergence of iterations of order-statistics filters are proposed.

Definition 2: A sequence of finite sets $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$ *converges* to set Y ($X_n \rightarrow Y$) if $X_n = Y$, for every $n \geq N$ for some $N \in \mathcal{N}$.

Definition 3: A sequence of sets $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$ is *increasing* (resp., *decreasing*) if $X_n \subseteq X_{n+1}$ (resp., $X_{n+1} \subseteq X_n$), for every $n \geq N$ for some $N \in \mathcal{N}$.

Definition 4: A sequence of sets $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$ is *uniformly bounded* if there exists a finite set $B \subseteq \mathcal{Z}^n$ such that $X_n \subseteq B$, for every $n \geq N$ for some $N \in \mathcal{N}$.

Lemma 2 [15]: Given a sequence of finite sets $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$, a) if $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$ is decreasing or b) if $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$ is increasing and uniformly bounded; then, the sequence $\{X_n \subseteq \mathcal{Z}^n : n \in \mathcal{N}\}$ converges.

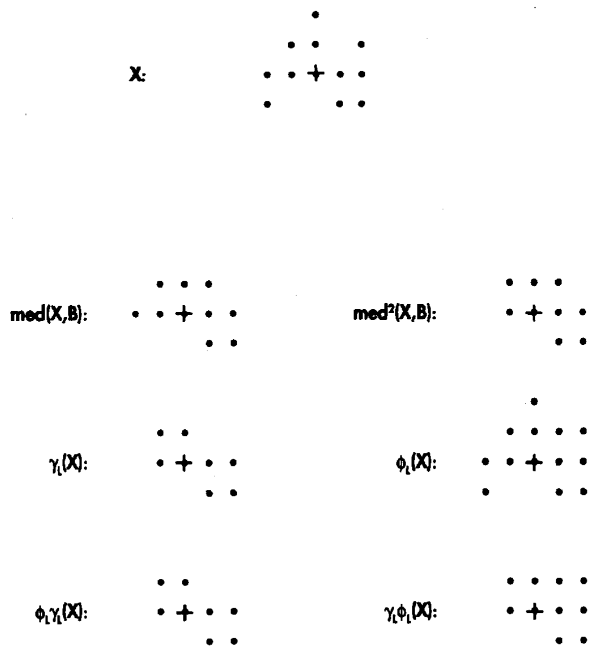

Fig. 3. Morphological representation of median filters (2-D signals): Morphological (lower and upper) bounds $\gamma_L(X)$ and $\phi_L(X)$ (respectively) on the median filter $med(X;B)$. Morphological (lower and upper) bounds $\phi_L\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ (respectively) on the (second) iteration of the median filter $med^2(X;B)$. Convergence of iterations of the median filter $\{med^n(X;B)\text{: } n \in \mathcal{N}\}$. The root $med^2(X;B)$ of the median filter with respect to window B. The roots $\phi_L(\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ of the median filter with respect to window B.

In the following proposition, we derive conditions for openings and closings to serve as bounds (lower and upper, respectively) on the convergence of iterations of order-statistics filters.[1]

Proposition 1: If there exist sets L and B such that $L \subseteq_m B$ (resp., $L \subseteq_{(|B|-m+1)} B$) whenever $m \geq (|B|+1)/2$ (resp., $m \leq (|B|+1)/2$), if X is finite, and if $m\text{th } OS^n(X;B) \to Y$, then

$$\gamma_L(X) \subseteq Y \subseteq \phi_L(X).$$

The morphological bounds derived in the previous proposition provide a restriction on the convergence of iterations of order-statistics filters.

In the following, we derive conditions for the convergence of iterations of order-statistics filters.[2,3]

Theorem 1: Given a finite set X, a) if $m\text{th } OS^{N+1}(X;B) \subseteq m\text{th } OS^N(X;B)$, for some $N \in \mathcal{N}$ or b) if $m\text{th } OS^N(X;B) \subseteq m\text{th } OS^{N+1}(X;B)$ for some $N \in \mathcal{N}$ and if there exist sets L and B such that $L \subseteq_{(|B|-m+1)} B$; then, the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges.

Corollary 1: Given a finite set X, a) if $m\text{th } OS^N(X;B)$ is L-closed, for some $N \in \mathcal{N}$ and if there exist sets L and B such that $L \subseteq_m B$ or b) if $m\text{th } OS^N(X;B)$ is L-open, for some $N \in \mathcal{N}$ and if there exist sets L and B such that $L \subseteq_{(|B|-m+1)} B$; then, the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges.

[1] The proof of the propositions are given in Appendix A.

[2] The proof of theorems are given in Appendix B.

[3] The proof of corollaries are given in Appendix C.

Corollary 2: Given a finite set X, a) if X is L-closed, and if there exist sets L and B such that $L \subseteq_m B$ or b) if X is L-open and if there exist sets L and B such that $L \subseteq_{(|B|-m+1)} B$; then, the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges.

Corollary 3: Given a finite set X, if X is convex, and if there exist sets L and B such that $L \subseteq_m B$, then the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges.

Example 2 (cont.): Consider a finite set $X \subseteq \mathcal{Z}$. It can be shown that the sequence $\{med^n(X;B) : n \in \mathcal{N}\}$ converges [4]. The convergence of iterations of the median filter $\{med^n(X;B) : n \in \mathcal{N}\}$ is depicted in Fig. 2.

Example 3 (cont.): Consider a finite set $X \subseteq \mathcal{Z}^2$ such that $med^2(X;B) \subseteq med(X;B)$. From Theorem 1(a) (with $N = 1$), we observe that the sequence $\{med^n(X;B) : n \in \mathcal{N}\}$ converges. The convergence of iterations of the median filter $\{med^n(X;B) : n \in \mathcal{N}\}$ is depicted in Fig. 3.

We have thus obtained conditions for the convergence of iterations of order-statistics filters. The convergence of iterations of order-statistics filters is represented by their roots. An investigation of the roots of order-statistics filters is provided in the next section.

V. Roots

In this section, we investigate the roots of order-statistics filters. Criteria for the morphological characterization of roots of order-statistics filters are proposed.

Definition 5: A set $X \subseteq \mathcal{Z}^n$ is a *root* of an mth OS filter with respect to window B if

$$m\text{th } OS(X;B) = X.$$

Let us use $\mathcal{L}$ and $\mathcal{L}^*$ to denote the subsets given by $\mathcal{L} = \{L \subseteq \mathcal{Z}^n$ and $\mathcal{L}^* = \{L \subseteq \mathcal{Z}^n$, respectively. Let us also use $\alpha(X)$ and $\varphi(X)$ to denote the opening and closing given by $\alpha(X) = \cup_{L\in\mathcal{L}}\gamma_L(X)$ and $\varphi(X) = \cap_{L\in\mathcal{L}^*}\phi_L(X)$, respectively.

In the following proposition, we derive conditions for open-closings and close-openings to serve as bounds (lower and upper, respectively) on the convergence of iterations of order-statistics filters.

Proposition 2: If there exist sets L and B such that $L \subseteq_m B$ (resp., $L \subseteq_{(|B|-m+1)} B$) whenever $m \geq (|B|+1)/2$ (resp., $m \leq (|B|+1)/2$), if X is finite, and if $m\text{th } OS^n(X;B) \to Y$, then

$$\varphi\alpha(X) \subseteq Y \subseteq \alpha\varphi(X).$$

The morphological bounds derived in the previous proposition provide a (tighter) restriction on the convergence of iterations (roots) of order-statistics filters (see also Proposition 1).

In the following, we derive conditions for the morphological characterization of roots of order-statistics filters.

Theorem 2: A set X is a root of the mth OS filter with respect to window B if and only if $\alpha(X) = X$ and $\varphi(X) = X$.

Corollary 4: If there exist sets L and B such that $L \subseteq_m B$ (resp., $L \subseteq_{(|B|-m+1)} B$) whenever $m \geq (|B|+1)/2$ (resp., $m \leq (|B|+1)/2$),

and if X is L-open and L-closed, then X is a root of the mth OS filter with respect to window B.

Example 2 (cont.): Consider a finite set $X \subseteq \mathcal{Z}$. It is easy to verify that $\alpha(X) = \gamma_L(X)$ and $\varphi(X) = \phi_L(X)$. From Proposition 2 (and the fact that $m\text{th } OS^n(X;B) \to Y$ [4]), we observe that $\phi_L(\gamma_L(X)) \subseteq Y \subseteq \gamma_L(\phi_L(X))$. From Theorem 2, we observe that X is a root of the median filter with respect to window B if and only if X is L-open and L-closed [4]. It is easy to verify that $\phi_L(\gamma_L(X))$ is L-

open and $\gamma_L(\phi_L(X))$ is L-closed. From Corollary 4, we observe that $\phi_L(\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ are roots of the median filter with respect to window B. The root $med^3(X;B)$ of the median filter with respect to window B is depicted in Fig. 2. The roots $\phi_L(\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ of the median filter with respect to window B are also depicted in Fig. 2.

Example 3 (cont.): Consider a set $X \subseteq \mathcal{Z}^2$ such that $med^2(X;B)$ is L-open and L-closed. From Corollary 4, we observe that $med^2(X;B)$ is a root of the median filter with respect to window B. Consider a set X such that $\phi_L(\gamma_L(X))$ is L-open and $\gamma_L(\phi_L(X))$ is L-closed. From Corollary 4, we observe that $\phi_L(\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ are roots of the median filter with respect to window B. The root $med^2(X;B)$ of the median filter with respect to window B is depicted in Fig. 3. The roots $\phi_L(\gamma_L(X))$ and $\gamma_L(\phi_L(X))$ of the median filter with respect to window B are also depicted in Fig. 3.

We have thus established a morphological characterization of the roots of order-statistics filters. The roots of order-statistics filters provide a representation of the class of signals (images) obtained by the convergence of their iterations.

VI. Summary

In this correspondence, a comprehensive theory of the convergence and characterization of roots of order-statistics filters has been proposed. Conditions for the convergence of iterations of order-statistics filters were proposed. Criteria for the morphological characterization of roots of order-statistics filters were also proposed. The results presented in this correspondence can be easily extended to gray-level signals. An extension of the approach presented in this correspondence to (arbitrary) nonlinear filters in a lattice theory framework appears in [15].

Appendix A
Proof of Propositions

Proof of Proposition 1: The proof is obtained directly from Property 3. □

Proof of Proposition 2: From Proposition 1, we observe that $\alpha(X) \subseteq Y \subseteq \varphi(X)$. It can also be shown that $\alpha(Y) = Y$, and $\varphi(Y) = Y$ (see Theorem 2). From the fact that the opening $\alpha(X)$ and the closing $\varphi(X)$ are increasing operators, we finally observe that $\varphi\alpha(X) \subseteq \varphi(Y) = Y = \alpha(Y) \subseteq \alpha\varphi(X)$. □

Appendix B
Proof of Theorems

Proof of Theorem 1: From the fact that X is finite, we observe that $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ is a sequence of finite sets. From Property 2, the fact that the order-statistics filter is an increasing operator, and the fact that mth $OS^{N+1}(X;B) \subseteq m$th $OS^N(X;B)$ (resp., mth $OS^N(X;B) \subseteq m$th $OS^{N+1}(X;B)$), we also observe that the sequence $\{m\text{th } OS^n(X;B) : n \geq N\}$ is decreasing (resp., increasing and uniformly bounded (by a finite set $\phi_L(X)$)). From Lemma 2, we finally observe that the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges. □

Proof of Theorem 2: Assume that $\alpha(X) = X$, and $\varphi(X) = X$. From Property 3, we observe that $\alpha(X) \subseteq m\text{th } OS(X;B) \subseteq \varphi(X)$. Therefore, we observe that $m\text{th } OS(X;B) = X$, i.e., X is a root of the mth OS filter with respect to window B.

Assume that X is a root of the mth OS filter with respect to window B. Therefore, we observe that $X \subseteq m\text{th } OS(X;B)$. From Lemma 1, we observe that $X \subseteq_m B$, i.e., $X \in \mathcal{L}$. From the fact that X is X-open, we observe that $X = \gamma_X(X) \subseteq \cup_{L \in \mathcal{L}} \gamma_L(X) = \alpha(X)$. From the antiextensivity of the opening $\alpha(X)$, we also observe that $\alpha(X) \subseteq X$. Therefore, we have $\alpha(X) = X$. By a similar argument, we have $\varphi(X) = X$. □

Appendix C
Proof of Corollaries

Proof of Corollary 1: From Properties 1 and 2 (with $X \rightarrow m\text{th } OS^N(X;B)$) and the fact that mth $OS^N(X;B)$ is L-closed (resp., L-open), we observe that mth $OS^{N+1}(X;B) \subseteq \phi_L(m\text{th} OS^N(X;B)) = m\text{th } OS^N(X;B)$ (resp., mth $OS^N(X;B) = \gamma_L(m\text{th } OS^N(X;B)) \subseteq m\text{th } OS^{N+1}(X;B)$). From Theorem 1, we observe that the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges. □

Proof of Corollary 2: The proof is obtained directly from Corollary 1 (with $N = 0$). □

Proof of Corollary 3: From the fact that a convex set is C-closed for any finite set C, we observe that X is L-closed [11]. From Corollary 2(a), we observe that the sequence $\{m\text{th } OS^n(X;B) : n \in \mathcal{N}\}$ converges. □

Proof of Corollary 4: The proof is obtained directly from Theorem 2. □

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