Correspondence

On the Invertibility of the Morphological Representation of Binary Images

Mohammed Charif-Chefchaouni and Dan Schonfeld

Abstract— **In this correspondence, we investigate the invertibility of the morphological representation of binary images. A criteria for the invertibility of the morphological representation of binary images is proposed. Necessary and sufficient conditions for the exact reconstruction of a binary image from its morphological representation are provided.**

I. Introduction

Many image analysis applications rely on the existence of image representation schemes that characterize an image in a compressed form [1]–[9]. The applicability of the image representation scheme is often dependent on the invertibility of the resulting representation (e.g., image coding [3]–[4]). Invertibility, therefore, poses a constraint on the compressibility of the image representation scheme.

Numerous morphological image representation schemes have been developed over the past decade [1]–[9]. An important class of morphological representation schemes, known as the morphological image representation, has been proposed by Goutsias and Schonfeld [8]–[9]. The morphological representation of an image is defined in terms of a morphological thining transformation. Goutsias and Schonfeld [8]–[9] have also proposed sufficient conditions, given by a direct restriction on the morphological thining transformation, for the exact reconstruction of a binary image from its morphological representation.

In this correspondence, we investigate the invertibility of the morphological representation of binary images. In Section II, some basic notions of mathematical morphology are introduced. In Section III, the morphological representation of binary images is presented. A criteria for the invertibility of the morphological representation of binary images is also proposed. In Section IV, conditions for the invertibility of the morphological representation of binary images are derived. We provide necessary and sufficient conditions for the exact reconstruction of a binary image from its morphological representation. The necessary and sufficient conditions obtained are given by a restriction defined in terms of the morphological thining transformation. The necessary and sufficient conditions, however, do not form a direct constraint on the morphological thining transformation. We subsequently use the necessary and sufficient conditions to derive sufficient conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of a binary image from its morphological representation. The sufficient conditions obtained are identical to the sufficient conditions proposed in [8] and [9]. We consequently provide an alternative proof of the sufficient conditions. We also derive necessary conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of a binary image from its morphological representation. The necessary conditions derived are obtained by two separate approaches. In the first approach, we use the necessary and sufficient conditions to derive necessary conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of a binary image from its morphological representation. In the second approach, however, we use the morphological reconstruction transformation directly to derive necessary conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of a binary image from its morphological representation. Finally, in Section V, a brief summary and discussion of our results is provided.

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The authors are with the Signal and Image Research Laboratory, Department of Electrical Engineering and Computer Science, University of Illinois at Chicago, Chicago, IL 60680-4348.

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II. Preliminaries

In this section, we introduce some basic notions of mathematical morphology.

We shall restrict ourselves in this presentation to discrete and binary images $\{f(x) \in \{0,1\}\colon\ x \in \mathcal{Z}^2\}$ (where $\mathcal{Z}$ denotes the set of integers). A discrete and binary image will be denoted by a set X given by $X = \{x \in \mathcal{Z}^2\colon\ f(x) = 1\}$. The complement X^c of a set X is given by $X^c = \{x \in \mathcal{Z}^2\colon\ f(x) = 0\}$. The symmetric $\check{X}$ of a set X is given by $\check{X} = \{-x\colon\ x \in X\}$. The translation X_y of a set X by vector y is given by $X_y = \{z\colon\ z = x+y, x \in X\}$. The set difference $X_1 - X_2$ of sets X_1 and X_2 is given by $X_1 - X_2 = X_1 \cap X_2^c$. The cardinality $|X|$ of a set X is given by $|X| = \Sigma_{x\in\mathcal{Z}^2}\ f(x)$, i.e., the cardinality of a set denotes the total number of elements contained in the set.

Let the structuring element B denote a finite set (i.e., $|B| < \infty$). The erosion $X \ominus B$ and dilation $X \oplus B$ of a set X by a structuring element B are given by

$$X \ominus B = \cap_{b\in\check{B}}\ X_b \tag{1a}$$

and

$$X \oplus B = \cup_{b\in\check{B}}\ X_b \tag{1b}$$

respectively; whereas the opening $\gamma_B(X)$ and closing $\phi_B(X)$ of a set X by a structuring element B are given by

$$\gamma_B(X) = X \ominus B \oplus \check{B} \tag{2a}$$

and

$$\phi_B(X) = X \oplus B \ominus \check{B} \tag{2b}$$

respectively.

III. Morphological Image Representation

In this section, we present the morphological representation of binary images. A criteria for the invertibility of the morphological representation of binary images is also proposed.

Consider a sequence of structuring elements $\{B(n)\colon\ n \geq 0\}$ such that $(0,0) \in B(n)$, for all n, and $B(n) \neq \{(0,0)\}$. The sequence $A(n)$ is given by $A(n+1) = A(n) \oplus B(n)$, for $n \geq 0$, and $A(0) = \{(0,0)\}$. Finally, the integer N is given by $N = \max\{n\colon\ X \ominus A(n) \neq \emptyset\}$.

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Let the morphological thining transformations $\{\Psi_n[\bullet]\colon n = 0, 1, \cdots, N\}$ denote a collection of morphological transformations. The morphological representation $R(X)$ of image X is given by

$$R(X) = \{R_0(X), R_1(X), \cdots, R_N(X)\} \tag{3}$$

where the morphological representation subset of order n $R_n(X)$ is given by

$$R_n(X) = X \ominus A(n) - \Psi_n(X \ominus A(n)) \tag{4}$$

for $n = 0, 1, \cdots, N$.

In a general framework, a morphological representation $R(X)$ is invertible if there exists a transformation $R^{-1}(\bullet)$ such that

$$X = R^{-1}(R(X)). \tag{5}$$

In this presentation, however, we shall restrict ourselves to a particular reconstruction criteria.

A morphological representation $R(X)$ of an image X is invertible if

$$\gamma_{A(n)}(X) = \cup_{m=n}^{N}[R_m(X) \oplus A(m)] \tag{6}$$

for $n = 0, 1, \cdots, N$, i.e., the invertibility criterion used in this presentation requires the exact reconstruction of the morphological openings $\gamma_{A(n)}(X)$, for all $n = 0, 1, \cdots, N$, from the morphological representation subsets $\{R_m(X)\colon m = n, n+1, \cdots, N\}$.

It is important to notice in particular that, if a morphological representation $R(X)$ of an image X is invertible, then

$$X = \cup_{m=0}^{N}[R_m(X) \oplus A(m)] \tag{7}$$

i.e., the invertibility criterion used in this presentation also requires the exact reconstruction of the image X from the morphological representation $R(X)$.

IV. Morphological Image Reconstruction

In this section, we derive conditions for the invertibility of the morphological representation of binary images.

A. Necessary and Sufficient Conditions

We shall now provide necessary and sufficient conditions for the invertibility of the morphological representation of binary images.

In the following theorem, we derive necessary and sufficient conditions for the exact reconstruction of the morphological openings $\gamma_{A(n)}(X)$, for all $n = 0, 1, \cdots, N$, from the morphological representation subsets $\{R_m(X)\colon m = n, n+1, \cdots, N\}$.

Theorem 1: We have

$$\gamma_{A(n)}(X) = \cup_{m=n}^{N}[R_m(X) \oplus A(m)] \tag{8}$$

for all $n = 0, 1, \cdots, N$, if and only if

$$\begin{aligned}\gamma_{A(n)}(X) &- \gamma_{A(n+1)}(X)\\ &\subseteq [X \ominus A(n) - \Psi_n(X \ominus A(n))] \oplus A(n)\end{aligned} \tag{9}$$

for all $n = 0, 1, \cdots, N$.

Proof: We assume that (8) is satisfied. From (8), we observe that

$$\begin{aligned}\gamma_{A(n)}(X) &= \cup_{m=n}^{N}\,[R_m(X) \oplus A(m)]\\ &= \cup_{m=n+1}^{N}\,[R_m(X) \oplus A(m)]\\ &\quad \cup [R_n(X) \oplus A(n)]\\ &= \gamma_{A(n+1)}(X) \cup [R_n(X) \oplus A(n)].\end{aligned} \tag{10}$$

From (10), we observe that

$$\begin{aligned}\gamma_{A(n)}(X) &- \gamma_{A(n+1)}(X)\\ &= [\gamma_{A(n+1)}(X) \cup [R_n(X) \oplus A(n)]] - \gamma_{A(n+1)}(X)\\ &\subseteq [R_n(X) \oplus A(n)].\end{aligned} \tag{11}$$

Finally, from (4) and (11), we obtain (9).

We now assume that (9) is satisfied. From (4) and (9), we observe that

$$\begin{aligned}\gamma_{A(n)}(X) &\subseteq \gamma_{A(n+1)}(X) \cup [\gamma_{A(n)}(X) - \gamma_{A(n+1)}(X)]\\ &\subseteq \gamma_{A(n+1)}(X) \cup [R_n(X) \oplus A(n)]\\ &\subseteq \cdots \subseteq \gamma_{A(k+1)}(X) \cup [\cup_{m=n}^{k}[R_m(X) \oplus A(m)]]\\ &\subseteq \cdots \subseteq \cup_{m=n}^{N}[R_m(X) \oplus A(m)].\end{aligned} \tag{12}$$

From (4) we observe that $\gamma_{A(m)}(X) \supseteq [R_m(X) \oplus A(m)]$, for $m = n, n+1, \cdots, N$. Therefore, we observe that

$$\gamma_{A(n)}(X) = \cup_{m=n}^{N}\,\gamma_{A(m)}(X) \supseteq \cup_{m=n}^{N}[R_m(X) \oplus A(m)]. \tag{13}$$

Finally, from (12) and (13), we obtain (8).

This completes the proof. □

The necessary and sufficient conditions obtained are given by a restriction defined in terms of the morphological thining transformation (sec. (9)). The necessary and sufficient conditions, however, do not form a direct constraint on the morphological thining transformation. In the next two subsections, we provide conditions given by a direct restriction on the morphological thining transformation for the invertibility of the morphological representation of binary images.

B. Sufficient Conditions

We shall now use the necessary and sufficient conditions to provide sufficient conditions given by a direct restriction on the morphological thining transformation for the invertibility of the morphological representation of binary images.

The following lemma shall be used in the proof of Corollary 1.

Lemma 1: Given X_1, X_2, and B, then

$$(X_1 \oplus B) - (X_2 \oplus B) \subseteq (X_1 - X_2) \oplus B. \tag{14}$$

Proof: Let us consider $x \in (X_1 \oplus B - X_2 \oplus B)$. Therefore, $x = \alpha + b$, where $\alpha \in X_1$ and $b \in B$. Moreover, $\alpha \notin X_2$. Finally, $x \in (X_1 - X_2) \oplus B$.

This completes the proof. □

In the following corollary, we derive sufficient conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of the morphological openings $\gamma_{A(n)}(X)$, for all $n = 0, 1, \cdots, N$, from the morphological representation subsets $\{R_m(X)\colon m = n, n+1, \cdots, N\}$.

Corollary 1: If

$$\Psi_n(X \ominus A(n)) \oplus A(n) \subseteq \gamma_{A(n+1)}(X) \tag{15}$$

for all $n = 0, 1, \cdots, N$, then

$$\gamma_{A(n)}(X) = \cup_{m=n}^{N}[R_m(X) \oplus A(m)] \tag{16}$$

for all $n = 0, 1, \cdots, N$.

Proof: From (14) and (15), we observe that

$$\begin{aligned}\gamma_{A(n)}(X) &- \gamma_{A(n+1)}(X)\\ &\subseteq \gamma_{A(n)}(X) - \Psi_n(X \ominus A(n)) \oplus A(n)\\ &= (X \ominus A(n)) \oplus A(n) - \Psi_n(X \ominus A(n)) \oplus A(n)\\ &\subseteq [X \ominus A(n) - \Psi_n(X \ominus A(n))] \oplus A(n).\end{aligned} \tag{17}$$

This completes the proof (see Theorem 1). □

The sufficient conditions presented in Corollary 1 are identical to the sufficient conditions proposed in [8] and [9]. An alternative proof of the sufficient conditions has consequently been provided.

C. Necessary Conditions

We shall now use two separate approaches to provide necessary conditions given by a direct restriction on the morphological thining transformation for the invertibility of the morphological representation of binary images.

In the following corollary, we use the necessary and sufficient conditions to derive necessary conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of the morphological openings $\gamma_{A(n)}(X)$, for all $n = 0, 1, \cdots, N$, from the morphological representation subsets $\{R_m(X)\colon\ m = n, n+1, \cdots, N\}$.

Corollary 2: If

$$\gamma_{A(n)}(X) = \cup_{m=n}^{N}[R_m(X) \oplus A(m)] \tag{18}$$

for all $n = 0, 1, \cdots, N$, then

$$\Psi_n(X \ominus A(n)) \ominus A(n) \subseteq \gamma_{A(n+1)}(X) \cup \phi_{A(n)}(X^c) \tag{19}$$

for all $n = 0, 1, \cdots, N$.

Proof: From (9) (see Theorem 1) and the fact that $(X_1 \cap X_2) \oplus B \subseteq (X_1 \oplus B) \cap (X_2 \oplus B)$ [5], we observe that

$$\begin{aligned}\gamma_{A(n)}&(X) - \gamma_{A(n+1)}(X)\\ &\subseteq [X \ominus A(n) - \Psi_n(X \ominus A(n)] \oplus A(n)\\ &= [[X \ominus A(n)] \cap [\Psi_n(X \ominus A(n))^c] \oplus A(n)\\ &\subseteq [X \ominus A(n) \oplus A(n)] \cap [[\Psi_n(X \ominus A(n))]^c \oplus A(n)]\\ &= \gamma_{A(n)}(X) \cap [\Psi_n(X \ominus A(n)) \ominus A(n)]^c.\\ &= \gamma_{A(n)}(X) - [\Psi_n(X \ominus A(n)) \ominus A(n)].\end{aligned} \tag{20}$$

From (20), we observe that

$$\begin{aligned}\Psi_n(X \ominus A(n)) \ominus A(n) &\subseteq \gamma_{A(n+1)}(X) \cup [\gamma_{A(n)}(X)]^c\\ &= \gamma_{A(n+1)}(X) \cup \phi_{A(n)}(X^c).\end{aligned} \tag{21}$$

This completes the proof. □

The necessary conditions presented in Corollary 2 impose a direct constraint on all morphological thining transformations.

In the following corollary, we use the morphological reconstruction transformation directly to derive necessary conditions given by a direct restriction on the morphological thining transformation for the exact reconstruction of the morphological openings $\gamma_{A(n)}(X)$, for all $n = 0, 1, \cdots, N$, from the morphological representation subsets $\{R_m(X)\colon\ m = n, n+1, \cdots, N\}$.

Corollary 3: If

$$\gamma_{A(n)}(X) = \cup_{m=n}^{N}[R_m(X) \oplus A(m)] \tag{22}$$

for any $n = 0, 1, \cdots, N$, then

$$\Psi_l(X \ominus A(l)) \oplus A(l) \not\supset \gamma_{A(l+1)}(X) \tag{23}$$

for some $l = n, n+1, \cdots, N$.

Proof: Let us assume (by contradiction) that

$$\Psi_l(X \ominus A(l)) \oplus A(l) \supset \gamma_{A(l+1)}(X) \tag{24}$$

for all $l = n, n+1, \cdots, N$. From (4) and (24) and the fact that $(X_1 \cup X_2) \oplus B = (X_1 \oplus B) \cup (X_2 \oplus B)$ [5], we observe that

$$\begin{aligned}\gamma_{A(n)}(X) &= [R_n(X) \cup \Psi_n(X \ominus A(n))] \oplus A(n)\\ &= [R_n(X) \oplus A(n)] \cup [\Psi_n(X \ominus A(n)) \oplus A(n)]\\ &\supset [R_n(X) \oplus A(n)] \cup [\gamma_{A(n+1}(X)]\\ &\supset \cdots \supset \cup_{m=n}^{k}[R_m(X) \oplus A(m)] \cup [\gamma_{A(k+1)}(X)]\\ &\supset \cdots \supset \cup_{m=n}^{N}[R_m(X) \oplus A(m)].\end{aligned} \tag{25}$$

This completes the proof. □

The necessary conditions presented in Corollary 3 yield a direct constraint on a specific morphological thining transformations.

V. Summary

In this correspondence, we investigated the invertibility of the morphological representation of binary images. A criteria for the invertibility of the morphological representation of binary images has been proposed. Necessary and sufficient conditions for the exact reconstruction of a binary image from its morphological representation were provided. The necessary and sufficient conditions obtained are given by a restriction defined in terms of the morphological thining transformation. The necessary and sufficient conditions, however, do not form a direct constraint on the morphological thining transformation. The necessary and sufficient conditions were subsequently used to derive sufficient conditions, which were given by a direct restriction on the morphological thining transformation for the exact reconstruction of a binary image from its morphological representation. The sufficient conditions obtained are identical to the sufficient conditions proposed in [8] and [9]. An alternative proof of the sufficient conditions has consequently been provided. Necessary conditions, which were given by a direct restriction on the morphological thining transformation for the exact reconstruction of a binary image from its morphological representation were also derived. The necessary conditions derived were obtained by two separate approaches. In the first approach, the necessary and sufficient conditions were used to derive necessary conditions, which were given by a direct restriction on the morphological thining transformation, for the exact reconstruction of a binary image from its morphological representation. In the second approach, however, the morphological reconstruction transformation directly was used to derive necessary conditions, which were given by a direct restriction on the morphological thining transformation, for the exact reconstruction of a binary image from its morphological representation.

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