Inverse Compute-and-Forward: Extracting Messages from Simultaneously Transmitted Equations

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Compute and forward

[Nazer, Gastpar, 2011]
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Key idea

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \]

\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \]

\[ \Rightarrow \hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2 \]
Key idea

Option 1: transmit equations

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \]
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$\mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2$

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$\Rightarrow \mathbf{\hat{w}}_1, \mathbf{\hat{w}}_2$

(if matrix $A$ invertible)
Key idea

Option 1: transmit equations

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(if matrix A invertible)

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Option 2: extract directly over the air!
Key idea

Option 1: transmit equations

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \]
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Option 2: extract directly over the air!

(allowable equations or MAC with common messages)

(if matrix A invertible)
Combine for a unified rate region!

Inverse Compute and forward to extract messages

Compute and forward to decode $aw_1 \oplus bw_2$
Outline
Outline

• Problem statement
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• Approach 1: allowable equations
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• Approach 1: allowable equations

• Approach 2: MAC with common messages
Outline

• Problem statement

• Approach 1: allowable equations

• Approach 2: MAC with common messages

• Beyond 2 users
Outline

- Problem statement
- Approach 1: allowable equations
- Approach 2: MAC with common messages
- Beyond 2 users
- Case study
Problem statement

\[ w_1 \in \mathbb{F}_q^{k_1}, \text{ for } q \text{ prime, } k_1 = \frac{nR_1}{\log_2 q} \]

\[ w_2 \in \mathbb{F}_q^{k_2}, \text{ for } q \text{ prime, } k_2 = \frac{nR_2}{\log_2 q} \]

(zero-pad)
Problem statement

\[ w_1 \in \mathbb{F}_q^{k_1}, \text{ for } q \text{ prime, } k_1 = \frac{nR_1}{\log_2 q} \]

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \]

\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

\[ w_2 \in \mathbb{F}_q^{k_2}, \text{ for } q \text{ prime, } k_2 = \frac{nR_2}{\log_2 q} \]
Problem statement

\[ w_1 \in \mathbb{F}_q^{k_1}, \text{ for } q \text{ prime}, \quad k_1 = \frac{nR_1}{\log_2 q} \]

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \]

\[ u_2 = a_{21}w_1 \oplus a_{22}w_2 \]

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \] is full rank over \( \mathbb{F}_q \).

\[ \oplus \text{ denotes finite field addition} \]

\[ w_2 \in \mathbb{F}_q^{k_2}, \text{ for } q \text{ prime}, \quad k_2 = \frac{nR_2}{\log_2 q} \]
Problem statement

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \]

\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \]

(wzero-pad)

\[ \mathbf{w}_1 \text{ rate } R_1 \]

\[ \mathbf{w}_2 \text{ rate } R_2 \]
Problem statement

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \]

\[ \Rightarrow X_1 \]

Power \( S_1 \)

\[ \Rightarrow X_2 \]

Power \( S_2 \)

\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \]
Problem statement

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \]

\[ \Rightarrow X_1 \]

Power \( S_1 \)

\[ \Rightarrow X_2 \]

Power \( S_2 \)

\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

\[ Y = X_1 + X_2 + Z \]

\[ Z \sim \mathcal{N}(0, 1) \]

\( \text{(zero-pad)} \)

<table>
<thead>
<tr>
<th>( w_1 ) rate ( R_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_2 ) rate ( R_2 )</td>
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</tbody>
</table>
Problem statement

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \]
\[ \Rightarrow X_1 \quad \text{Power } S_1 \]
\[ \Rightarrow X_2 \quad \text{Power } S_2 \]
\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \]

\[ Y = X_1 + X_2 + Z \]
\[ Z \sim \mathcal{N}(0, 1) \]
\[ \Rightarrow \hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2 \]
\[ \Pr((\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2) \neq (\mathbf{w}_1, \mathbf{w}_2)) < \epsilon. \]
Outline

• Problem statement

• Approach 1: allowable equations

• Approach 2: MAC with common messages

• Beyond 2 users

• Case study
Goal - derive ICF rate region
Goal - derive ICF rate region

ICF rate region

CF rate region
Goal - derive ICF rate region

Intersection of CF and ICF regions

CF rate region

ICF rate region

Time sharing

$R_2$

$R_1$
Goal - derive ICF rate region

- Approach 1: allowable equations (independent messages at Txs)
Goal - derive ICF rate region

- Approach 1: allowable equations (independent messages at Txs)
- Approach 2: MAC with common messages (correlated messages at Txs)
Approach 1: allowable equations

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \]

\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

\[ \Rightarrow \hat{w}_1, \hat{w}_2 \]
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\[ |u_1| = |u_2| = 2^{n R_{MAX}} \]
Approach 1: allowable equations

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\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \]

\[ |\mathbf{u}_1| = |\mathbf{u}_2| = 2^{nR_{MAX}} \]

\[ 2R_{MAX} \geq R_1 + R_2 \]
Approach 1: allowable equations

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \]

Key idea: if one equation is fixed, limits the number of possibilities of the other!

\[ u_2 = a_{21}w_1 \oplus a_{22}w_2 \]

- \( w_1 \) rate \( R_1 \)
- \( w_2 \) rate \( R_2 \)

\[ |u_1| = |u_2| = 2^{nR_{MAX}} \quad \Rightarrow \quad 2R_{MAX} \geq R_1 + R_2 \]
Approach 1: allowable equations

- Generate $2^{nR_{\text{MAX}}}$ codewords of length $n$, $X_1^n$ i.i.d. $\sim \mathcal{N}(0, S_1)$.
- Generate $2^{nR_{\text{MAX}}}$ independent codewords $X_2^n$ i.i.d. $\sim \mathcal{N}(0, S_2)$.
- Transmit $X_1^n(u_1)$ and $X_2^n(u_2)$.
Approach 1: allowable equations

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- Transmit $X_1^n(u_1)$ and $X_2^n(u_2)$.
- Receive $Y^n = X_1^n(u_1) + X_2^n(u_2) + Z^n$ and decode $(\hat{u}_1, \hat{u}_2)$ such that $(X_1^n(\hat{u}_1), X_2^n(\hat{u}_2), Y^n)$ is jointly typical.
Approach 1: allowable equations

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- $P_e$ proceeds as in MAC channel EXCEPT that need to carefully count
  - $(u_1 = 0, u_2 \neq 0)$
  - $(u_1 \neq 0, u_2 = 0)$
  - $(u_1 \neq 0, u_2 \neq 0)$
  - $|\mathcal{M}_A(U_2|0)|$
  - $|\mathcal{M}_A(U_1|0)|$
  - $|\mathcal{M}_A(U_1, U_2)|$
Approach 1: allowable equations

- Generate $2^{nR_{\text{MAX}}}$ codewords of length $n$, $X_1^n$ i.i.d. $\sim \mathcal{N}(0, S_1)$.
- Generate $2^{nR_{\text{MAX}}}$ independent codewords $X_2^n$ i.i.d. $\sim \mathcal{N}(0, S_2)$.
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$P_e$ proceeds as in MAC channel EXCEPT that need to carefully count

- $(u_1 = 0, u_2 \neq 0)$
- $(u_1 \neq 0, u_2 = 0)$
- $(u_1 \neq 0, u_2 \neq 0)$

Count the number of allowable equations!
Cardinality Lemma

\[ \mathcal{M}_A(U_1, U_2) = \left\{ (u_1, u_2) : u_1 = a_{11}w_1 + a_{12}w_2, \right. \]
\[ \left. \quad u_2 = a_{21}w_1 + a_{22}w_2, \text{ for some } w_1, w_2 \right\} . \]

\[ \mathcal{M}_A(U_1 | u_2) = \left\{ u_1 : u_1 = a_{11}w_1 + a_{12}w_2 \text{ for some } w_1, w_2 \right. \]
\[ \left. \quad \text{satisfying } u_2 = a_{21}w_1 + a_{22}w_2 \right\} , \]
Cardinality Lemma

\[ \mathcal{M}_A(U_1, U_2) = \left\{ (u_1, u_2) : u_1 = a_{11}w_1 + a_{12}w_2, \right. \]
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\[ \mathcal{M}_A(U_1|u_2) = \left\{ u_1 : u_1 = a_{11}w_1 + a_{12}w_2 \text{ for some } w_1, w_2 \right. \]
\[ \left. \text{satisfying } u_2 = a_{21}w_1 + a_{22}w_2 \right\} , \]

**Cardinality lemma.** The set of allowable equations \( \mathcal{M}_A(U_1, U_2) \) and the set of conditionally allowable equations \( \mathcal{M}_A(U_\ell|u_m) \) have the following cardinalities:

\[ |\mathcal{M}_A(U_1, U_2)| = 2^{n(R_1 + R_2)} \]
\[ |\mathcal{M}_A(U_1|u_2)| = 2^{nR_{\text{MIN}}} \]
\[ |\mathcal{M}_A(U_2|u_1)| = 2^{nR_{\text{MIN}}} . \]
Proof of Cardinality Lemma

\[ |M_A(U_1, U_2)| = 2^{n(R_1 + R_2)} \text{ as } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \]
Proof of Cardinality Lemma

\[ |\mathcal{M}_A(U_1, U_2)| = 2^{n(R_1 + R_2)} \text{ as } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \]

\[ |\mathcal{M}_A(U_1|u_2)| = 2^{n \min(R_1, R_2)}. \text{ Assume WLOG } R_1 > R_2. \]

Each \( w_2 \) has one \( w_1 \) such that \( a_{21}w_1 + a_{22}w_2 = u_2 \).

Since \( u_1 \perp u_2 \Rightarrow 2^{nR_2} \) solutions.

(zero-pad)

\[ \begin{array}{cccccc}
| & | & | & | & | \\
\text{w}_1 \text{ rate } R_1 & & & & \end{array} \quad \begin{array}{cccc}
| & | & | & |
\text{w}_2 \text{ rate } R_2 & & & & \\
\end{array} \]
Approach 1: allowable equations

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \]

\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

\[ \min(R_1, R_2) < \min(C(S_1), C(S_2)) \]

\[ R_1 + R_2 < C(S_1 + S_2). \]

\[
P_e \leq \epsilon + |M_A(U_1|0)|2^{-n(I(X_1;Y|X_2)-\epsilon)} + |M_A(U_2|0)|2^{-n(I(X_2;Y|X_1)-\epsilon)} + |M_A(U_1, U_2)|2^{-n(I(X_1,X_2;Y)-\epsilon)}
= \epsilon + 2^{nR_{\text{MIN}}}2^{-n(I(X_1;Y|X_2)-\epsilon)} + 2^{nR_{\text{MIN}}}2^{-n(I(X_2;Y|X_1)-\epsilon)} + 2^n(R_1+R_2)2^{-n(I(X_1,X_2;Y)-\epsilon)}.\]
What if coefficients are zero?

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \]
\[ u_2 = a_{21}w_1 \oplus a_{22}w_2 \]

\[ R_{\text{min}} < I(X_1; Y|X_2) = C(S_1) \]
\[ R_{\text{min}} < I(X_2; Y|X_1) = C(S_2) \]
\[ R_1 + R_2 < I(X_1, X_2; Y) = C(S_1 + S_2). \]
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\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

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\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \]

\[ R_{\min} < I(X_1; Y | X_2) = C(S_1) \]
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\[ R_1 + R_2 < I(X_1, X_2; Y) = C(S_1 + S_2). \]
Outline

• Problem statement

• Approach 1: allowable equations

• Approach 2: MAC with common messages

• Beyond 2 users

• Case study
Approach 2: MAC with common messages

\[ w_1 \text{ rate } R_1 \]
\[ w_2 \text{ rate } R_2 \]
\[ \begin{array}{c}
\oplus \\
\end{array} \]
\[ \left\{ \begin{array}{c}
u_1 = a_{11}w_1 \oplus a_{12}w_2 \\
u_2 = a_{21}w_1 \oplus a_{22}w_2 \\
\end{array} \right. \]
Approach 2: MAC with common messages

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \]

\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

Common message rate \( R_0 \)
Approach 2: MAC with common messages

\[ w_1 \text{ rate } R_1 \]
\[ w_2 \text{ rate } R_2 \]
\[ w_1 \text{ rate } R_1 \]
\[ w_2 \text{ rate } R_2 \]

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \]
\[ u_2 = a_{21}w_1 \oplus a_{22}w_2 \]

Private message rate \( R_1 \)

Common message rate \( R_0 \)

[Slepian, Wolf, 1973], [Han, 1979]
Approach 2: MAC with common messages

\[
\begin{align*}
\mathbf{w}_1 \text{ rate } R_1 \\
\mathbf{w}_2 \text{ rate } R_2 \\
\mathbf{u}_1 &= a_{11}\mathbf{w}_1 \oplus a_{12}\mathbf{w}_2 \\
\mathbf{u}_2 &= a_{21}\mathbf{w}_1 \oplus a_{22}\mathbf{w}_2 \\
\end{align*}
\]
Approach 2: MAC with common messages

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \]
\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \]

Decoding all parts yields \( w_1 \) and \( w_2 \)

[Slepian, Wolf, 1973], [Han, 1979]
Approach 2: MAC with common messages

$w_1$ rate $R_1$

$w_2$ rate $R_2$

Private message rate $R_1$

Private message rate $R_2$

Common message rate $R_0$

Decoding all parts yields $w_1$ and $w_2$

$R_1 < I(X_1; Y|X_2, V)$

$R_2 < I(X_2; Y|X_1, V)$

$R_1 + R_2 < I(X_1, X_2; Y|V)$

$R_0 + R_1 + R_2 < I(X_1, X_2; Y)$

for some $p_V(v)p_{X_1|V}(x_1|v)p_{X_2|V}(x_2|v)$. 

[Slepian, Wolf, 1973], [Han, 1979]
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Beyond 2 users - allowable equations

\[ u_1 = a_{11} w_1 \oplus a_{12} w_2 \oplus a_{13} w_3 \]

\[ u_2 = a_{21} w_1 \oplus a_{22} w_2 \oplus a_{23} w_3 \]

\[ u_3 = a_{31} w_1 \oplus a_{32} w_2 \oplus a_{33} w_3 \]

\[ \Rightarrow \hat{w}_1, \hat{w}_2, \hat{w}_3 \]
Beyond 2 users - allowable equations

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \oplus a_{13} \mathbf{w}_3 \]

\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \oplus a_{23} \mathbf{w}_3 \]

\[ \mathbf{u}_3 = a_{31} \mathbf{w}_1 \oplus a_{32} \mathbf{w}_2 \oplus a_{33} \mathbf{w}_3 \]

\[ |\mathcal{M}_\mathbf{A}(U_1, U_2, U_3)| = 2^n(R_1 + R_2 + R_3) \]

\[ |\mathcal{M}_\mathbf{A}(U_1, U_2 | u_3)| = |\mathcal{M}_\mathbf{A}(U_1, U_3 | u_2)| = |\mathcal{M}_\mathbf{A}(U_2, U_3 | u_1)| = 2^n(R_{\text{Mid}} + R_{\text{Min}}) \]

\[ |\mathcal{M}_\mathbf{A}(U_1 | u_2, u_3)| = |\mathcal{M}_\mathbf{A}(U_2 | u_1, u_3)| = |\mathcal{M}_\mathbf{A}(U_3 | u_1, u_2)| = 2^n R_{\text{Min}} \]
Beyond 2 users - allowable equations

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \oplus a_{13}w_3 \]

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\[ u_3 = a_{31}w_1 \oplus a_{32}w_2 \oplus a_{33}w_3 \]

Use in MAC Pe calculations

\[ |\mathcal{M}_A(U_1, U_2, U_3)| = 2^{n(R_1+R_2+R_3)} \]

\[ |\mathcal{M}_A(U_1, U_2|u_3)| = |\mathcal{M}_A(U_1, U_3|u_2)| = |\mathcal{M}_A(U_2, U_3|u_1)| = 2^{n(R_{\text{mid}}+R_{\text{min}})} \]

\[ |\mathcal{M}_A(U_1|u_2, u_3)| = |\mathcal{M}_A(U_2|u_1, u_3)| = |\mathcal{M}_A(U_3|u_1, u_2)| = 2^{nR_{\text{min}}} \]

Sunday, July 31, 2011
Beyond 2 users - allowable equations

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \oplus a_{13} \mathbf{w}_3 \]

\[ \mathbf{u}_2 = a_{21} \mathbf{w}_1 \oplus a_{22} \mathbf{w}_2 \oplus a_{23} \mathbf{w}_3 \]

\[ \mathbf{u}_3 = a_{31} \mathbf{w}_1 \oplus a_{32} \mathbf{w}_2 \oplus a_{33} \mathbf{w}_3 \]

\[ \Rightarrow \hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3 \]

\[ R_{\text{MIN}} \leq \min\{C(S_1), C(S_2), C(S_3)\} \]

\[ R_{\text{MIN}} + R_{\text{MID}} \leq \min\{C(S_1 + S_2), C(S_1 + S_3), C(S_2 + S_3)\} \]

\[ R_{\text{MIN}} + R_{\text{MID}} + R_{\text{MAX}} \leq C(S_1 + S_2 + S_3) \]
Beyond 2 users - MAC with common messages

\[ \mathbf{u}_1 = a_{11} \mathbf{w}_1 \oplus a_{12} \mathbf{w}_2 \oplus a_{13} \mathbf{w}_3 \]

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Beyond 2 users - MAC with common messages

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Common message
Beyond 2 users - MAC with common messages

Private message

$$u_1 = a_{11}w_1 \oplus a_{12}w_2 \oplus a_{13}w_3$$

Private message

$$u_2 = a_{21}w_1 \oplus a_{22}w_2 \oplus a_{23}w_3$$

Private message

$$u_3 = a_{31}w_1 \oplus a_{32}w_2 \oplus a_{33}w_3$$

Common message
Beyond 2 users - MAC with common messages

Private message

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \oplus a_{13}w_3 \]

Common message (1,2)

Private message

\[ u_2 = a_{21}w_1 \oplus a_{22}w_2 \oplus a_{23}w_3 \]

Private message

\[ u_3 = a_{31}w_1 \oplus a_{32}w_2 \oplus a_{33}w_3 \]

Common message

Sunday, July 31, 2011
Beyond 2 users - MAC with common messages

\[ u_1 = a_{11}w_1 \oplus a_{12}w_2 \oplus a_{13}w_3 \]

Private message

\[ u_2 = a_{21}w_1 \oplus a_{22}w_2 \oplus a_{23}w_3 \]

Common message (1,2)

Private message

\[ u_3 = a_{31}w_1 \oplus a_{32}w_2 \oplus a_{33}w_3 \]

Common message (2,3)

Common message
Beyond 2 users - Message and equation alignment

\[ u_1 = w_1 \oplus w_2 \oplus w_3 \]

\[ u_2 = w_1 \oplus w_2 \oplus 3w_3 \]

\[ u_3 = w_1 \oplus 2w_2 \oplus 3w_3 \]
Beyond 2 users - Message and equation alignment

\[ u_1 = w_1 \oplus w_2 \oplus w_3 \]

Sub-equation alignment

\[ u_2 = w_1 \oplus w_2 \oplus 3w_3 \]

\[ u_3 = w_1 \oplus 2w_2 \oplus 3w_3 \]

\[ S_1 \]

\[ S_2 \]

\[ S_3 \]
Beyond 2 users - Message and equation alignment

\[ u_1 = w_1 \oplus w_2 \oplus w_3 \]

\[ u_2 = w_1 \oplus w_2 \oplus 3w_3 \]

\[ u_3 = w_1 \oplus 2w_2 \oplus 3w_3 \]

\[ |\mathcal{M}_A(U_3|u_1, u_2)| = 2^{n \min(R_1, R_2)} \text{ instead of } 2^{nR_{\text{MIN}}} \]
Beyond 2 users - Message and equation alignment

\[ u_1 = w_1 \oplus w_2 \oplus w_3 \]
\[ u_2 = w_1 \oplus w_2 \oplus 3w_3 \]
\[ u_3 = w_1 \oplus 2w_2 \oplus 3w_3 \]

Changes region!

\[ |\mathcal{M}_A(U_3|u_1, u_2)| = 2^n \min(R_1, R_2) \text{ instead of } 2^n R_{\text{MIN}} \]
Outline

• Problem statement

• Approach 1: allowable equations

• Approach 2: MAC with common messages

• Beyond 2 users

• Case study
Case study: exploiting interference

\[ R_1 < \frac{1}{2} \log \left( \frac{1}{2} + S \right), \]
\[ R_2 < \frac{1}{2} \log \left( \frac{1}{2} + S \right), \]
\[ \min(R_1, R_2) < \min \left( \frac{1}{2} \log(1 + S_3), \frac{1}{2} \log(1 + S_4) \right) \]
\[ R_1 + R_2 < \frac{1}{2} \log(1 + S_3 + S_4). \]
Case study: exploiting interference

\[ R_1 < \frac{1}{2} \log \left( \frac{1}{2} + S \right), \]
\[ R_2 < \frac{1}{2} \log \left( \frac{1}{2} + S \right), \]
\[ \min(R_1, R_2) < \min \left( \frac{1}{2} \log(1 + S_3), \frac{1}{2} \log(1 + S_4) \right) \]
\[ R_1 + R_2 < \frac{1}{2} \log(1 + S_3 + S_4). \]

No interference

\[ R_1 < \min \left\{ \frac{1}{2} \log(1 + S), \frac{1}{2} \log(1 + S_3) \right\} \]
\[ R_2 < \min \left\{ \frac{1}{2} \log(1 + S), \frac{1}{2} \log(1 + S_4) \right\} \]
\[ R_1 + R_2 < \frac{1}{2} \log(1 + S_3 + S_4). \]
Conclusion

Combine for a unified rate region!

Inverse
Compute and forward to extract messages

Compute and forward to decode $aw_1 \oplus bw_2$