Lattice codes for Gaussian relay channels

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Y. Song, N. Devroye, ``List decoding for nested lattices and applications to relay channels,'' Allerton 2010.

Structured codes for Gaussian networks
Structured codes for Gaussian networks

→ decode the sum

$X_1(W_1) + X_2(W_2)$

General "decode the sum" → Compute-and Forward

[Nazer, Gastpar, Trans IT, 2011]
Random codes for Gaussian networks

• **have:** cooperation

Structured codes for Gaussian networks

• **have:** “decode the sum”

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• **missing:** “decode the sum”

---

• **missing:** cooperation
Lattice codes for Gaussian relay networks?

- demonstrated utility for **single-hop** networks:
  - AWGN channel [Erez, Zamir, Trans. IT, 2004]
  - AWGN broadcast channel [Zamir, Shamai, Erez, Trans. IT, 2002]
  - AWGN multiple-access [Nazer, Gastpar, TransIT 2011] and ``dirty” multiple-access channels [Philosof, Khisti, Erez, Zamir, ISIT 2007]
  - Distributed source coding [Krithivasan, Pradhan, TransIT 2009]
  - AWGN interference channel: interference decoding / interference alignment in K>2 interference channels [Bresler, Parekh, Tse, TransIT, 2010] [Sridharan, Jafarian, Jafar, Shamai, arXiv 2008]

What about multi-hop networks?
Lattice codes for Gaussian relay networks?

- demonstrated utility for **specific two-hop** networks
  
  - AWGN two-way relay channels  
    [Nazer, Gastpar, TransIT 2011]  
    [Wilson, Narayanan, Pfister, Sprintson, Trans. IT, 2010]  
    [Nam, Chung, Lee, Trans. IT, 2010]

- AWGN multi-way relay channels  
  [Gunduz, Yener, Goldsmith, Poor, arXiv 2010],  
  [Sezgin, Avestimehr, Khajehnejad, Hassibi, arXiv 2010][Kim, Smida, D, ISIT 2011]

- demonstrated utility for **specific multi-hop** networks
  
  - AWGN two-hop interference relay channel  
    [Mohajer, Diggavi, Fragouli, Tse, arxiv 2010]

  - finite-field multi-hop interference relay channel  
    [Jeon, Chung, arxiv 2011]
Are these techniques enough for general relay networks?

Missing “cooperation” - combining of direct and relayed links
General relay network theorems

- AWGN relay channel DF and CF schemes first considered in [Cover, El Gamal, 1979]
  
- DF extension to arbitrary # of relays and sources in [Xie, Kumar, 2004]

- CF extension to arbitrary # of relays and sources in
  [Kramer, Gastpar, Gupta, 2004], [Lim, Kim, El Gamal, Chung, 2010]

- Quantize-and-map scheme for arbitrary # of relays and sources in
  [Avestimehr, Diggavi, Tse, 2011] (finite gap)

- Noisy network coding [Lim, Kim, El Gamal, Chung, 2010] (finite gap)

- Lattice-based schemes?
  
  - Quantize-and-map extended to lattice codes in [Ozgur, Diggavi, 2011]

  - Compute-and-forward framework [Nazer, Gastpar, TransIT, 2011], [Niesen, Whiting, 2011]

All based on RANDOM coding
Lattice codes missing in?

- AWGN relay channel?

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\text{"Cooperation"}

Various links carry same message!
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- Two-way relay channel in presence of direct links?
Enabling lattice "Cooperation"

Lattice list decoder

Lattices achieve DF rate

Intersect 2 lists

Lattices in multi-source networks

Lattices achieve CF rate

lattices good source and channel codes, special Wyner-Ziv

combine "decode sum" and direct-link cooperation
Outline - enabling cooperation via lattices

- Lattice notation
- Lattice list decoder

- Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay

- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel

- Lattices achieve CF rate for AWGN relay
Lattice notation

- \( \Lambda = \{ \lambda = G \mathbf{i} : \mathbf{i} \in \mathbb{Z}^n \} \), \( G \) the generator matrix

- *lattice quantizer* of \( \Lambda \):
  \[
  Q(\mathbf{X}) = \arg \min_{\lambda \in \Lambda} ||\mathbf{X} - \lambda||
  \]

- \( \mathbf{x} \mod \Lambda := \mathbf{x} - Q(\mathbf{x}) \)

- *fundamental region* \( \mathcal{V} := \{ \mathbf{x} : Q(\mathbf{x}) = \mathbf{0} \} \) of volume \( V \)

- *second moment per dimension of a uniform distribution over* \( \mathcal{V} \):
  \[
  \sigma^2(\Lambda) := \frac{1}{V} \cdot \frac{1}{n} \int_{\mathcal{V}} ||\mathbf{x}||^2 d\mathbf{x}
  \]
Nested lattice codes

- Nested lattice pair: \( \Lambda \subseteq \Lambda_c \) (\( \Lambda \) is Rogers-good and Poltyrev-good, \( \Lambda_c \) is Poltyrev-good)

- The code book \( \mathcal{C} = \{ \Lambda_c \cap V(\Lambda) \} \) is used to achieve the capacity of AWGN channel
  \[ [\text{Erez+Zamir, Trans. IT, 2004}] \]

- Coding rate: \( R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log \frac{V(\Lambda)}{V(\Lambda_c)} \) arbitrary (\# of \( \star \))
Nested lattice chains

- \( \Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_K \) (\( \Lambda_1, \Lambda_2, \ldots, \Lambda_{K-1} \) are Rogers-good and Poltyrev-good, \( \Lambda_K \) is Poltyrev-good). The nesting rates between any pairs in the chain can attain any arbitrary values as the dimension \( n \to \infty \).

[Krithivasan, Pradhan, 2007] [Nam, Chung, Lee, TransIT, 2010]

- A “good” lattice chain with length 3 is used in our list decoding scheme:

\[
\Lambda \subseteq \Lambda_s \subseteq \Lambda_c
\]
Lattice list decoder

- IDEA: decode to \( \Lambda \) rather than to \( \Lambda_c \)

- results in a list of codewords

- require correct codeword to be in list

- how many (lower bound) are in list?
Encoding

- message of rate $R$ over the AWGN channel $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ subject to the average power constraint $P$

- **Encoding:** take $\mathbf{t} \in \mathcal{C}_{\Lambda_c,\mathcal{V}}$ associated with message of rate $R$ and $\mathbf{X} = (\mathbf{t} - \mathbf{U}) \mod \Lambda$

- $\mathbf{U}$ is a dither signal uniformly distributed over $\mathcal{V}$. 
Decoding

- Receiver first computes
  \[ Y' = (\alpha Y + U) \mod \Lambda \]
  \[ = (t - (1 - \alpha)X + \alpha Z) \mod \Lambda \]
  \[ = (t + Z') \mod \Lambda \]

- Receiver then decodes the list of codewords \( \hat{t} \):
  \[ L(\hat{t}) := S_{\mathcal{V}_s, \Lambda_c}(Y') \mod \Lambda \]

\[ S_{\mathcal{V}_s, \Lambda_c}(Y') = \{ \Lambda_c \cap (Y' + \mathcal{V}_s) \} \]

\[ \star = Y' \]
Lattice list decoder

- Probability of error for list decoding: \( P_e := \Pr \{ t \notin L(\hat{t}) \} \)

\[ \Lambda \subseteq \Lambda_s \subseteq \Lambda_c \]

- Same list as

\[ S_{\nu_s,\Lambda_c}(Y') = \{ \Lambda_c \cap (Y' + \nu_s) \} \]

\[ \star = Y' \]

- Easy to count \# in list

\[ Q_{\nu_s,\Lambda_c}(Y') = \bigcup_{\lambda_c \in \Lambda_c} \{ \lambda_c | Y' \in (\lambda_c + \nu_s) \} \]

- Easy to bound probability of error

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Lattice list decoder

• Theorem 1: Using the encoding and decoding scheme defined above, the receiver decodes a list of codewords of size $2^n(R-C(P/N))$ with probability of error $P_e \to 0$ as $n \to \infty$
Outline - enabling cooperation via lattices

- Lattice notation

- Lattice list decoder

- Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay

- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel

- Lattices achieve CF rate for AWGN relay
Decode and forward relaying

\[ R_{DF} = \max_{p(x_1, x_R)} \{ \min\{ I(X_1; Y_R | X_R), I(X_1, X_R; Y_2) \} \} \]

- Irregular Markov Encoding with Successive Decoding
  [Cover, El Gamal 1979]

- Regular Encoding with Backward Decoding
  [Willems 1992]

- Regular Encoding with Sliding Window Decoding
  [Xie, Kumar 2002]

- Nice survey
  [Kramer, Gastpar, Gupta 2005]
Single source: lattice DF

Lattices achieve the DF rate for the relay channel. The following Decode-and-Forward rates can be achieved using nested lattice codes for the Gaussian relay channel:

\[
R < \max_{\alpha \in [0,1]} \min \left\{ \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right), \frac{1}{2} \log \left( 1 + \frac{P + P_R + 2\sqrt{\alpha PP_R}}{N_2} \right) \right\}.
\]

Achieved using
NESTED LATTICE CODES!
Central idea behind using lists

- view cooperation between links as intersection of independent lists

$L_{1-2}(w) \cap L_{R-2}(w) \Rightarrow \text{UNIQUE } w$
An aside.....

• ideally would want this list, rather than forcing a decode.....
Mimic all steps with lattice codes

### Encoding

<table>
<thead>
<tr>
<th>Block 1</th>
<th>Block 2</th>
<th>Block 3</th>
<th>Block 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1(w_1, 1)$</td>
<td>$x_1(w_2, w_1)$</td>
<td>$x_1(w_3, w_2)$</td>
<td>$x_1(1, w_3)$</td>
</tr>
<tr>
<td>$x_R(1)$</td>
<td>$x_R(w_1)$</td>
<td>$x_R(w_2)$</td>
<td>$x_R(w_3)$</td>
</tr>
</tbody>
</table>

### Decoding

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{1-2}(w_1)$</td>
<td>$L_{R-2}(w_1)$</td>
<td>$L_{R-2}(w_2)$</td>
</tr>
<tr>
<td>$L_{1-2}(w_2)$</td>
<td>$L_{1-2}(w_3)$</td>
<td>$L_{1-2}(w_3)$</td>
</tr>
</tbody>
</table>
**Encoding**

**Block b**

\[ x_b(w_b, w_{b-1}) = x'_1(w_b) + x'_2(w_{b-1}) \]

\[ x_R(w_{b-1}) = x'_2(w_{b-1}) \]

\[ \sigma^2(\Lambda_1) = \alpha P \]

\[ \sigma^2(\Lambda_2) = \bar{\alpha} P \]

\[ \Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1} \]

\[ \Lambda_2 \subseteq \Lambda_{s2} \subseteq \Lambda_{c2} \]

**Decoding**

**Block b**

\[ \Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1} \]

\[ \Lambda_2 \subseteq \Lambda_{s2} \subseteq \Lambda_{c2} \]

**Intersection of independent lists**

\[ \Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1} \]

**Lattice decoder**
• At relay block $b$:

$$Y_R(b) = X'_1(w_b) + X'_2(w_{b-1}) + X_R(w_{b-1}) + Z_R$$

$$R < \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_R}\right)$$

• At destination block $b$:

$$Y_2(b) = X'_1(w_b) + X'_2(w_{b-1}) + X_R(w_{b-1}) + Z_2$$

$$= X'_1(w_b) + \left(1 + \sqrt{\frac{P_R}{\alpha P}}\right) X'_2(w_{b-1}) + Z_2$$

• Decodes $L_{R-2}(w_{b-1})$ of size $2^{n(R-R_R)}$:

$$R_R < \frac{1}{2} \log \left(1 + \frac{(\sqrt{\alpha P} + \sqrt{P_R})^2}{\alpha P + N_2}\right)$$

• intersect $L_{R-2}(w_{b-1})$ and $L_{1-2}(w_{b-1})$ from previous block

$$R < \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_2}\right) + R_R$$

$$< \frac{1}{2} \log \left(1 + \frac{P + P_R + 2\sqrt{\alpha PP_R}}{N_2}\right).$$

• subtract $X'_2(w_{b-1})$ and decode list $L_{1-2}(w_b)$ of size $2^{n(R-C(\alpha P/(N_2)))}$
Outline - enabling cooperation via lattices

• Lattice notation

• Lattice list decoder

• Single source DF applications:
  • Lattices achieve DF rate for AWGN relay
  • Lattices achieve DF rate for AWGN multi-relay

• Multi-source DF applications:
  • Lattices for two-way relay channel with direct links
  • Lattices for multiple-access relay channel

• Lattices achieve CF rate for AWGN relay
Single-source, multiple relay

\[
Y_2 = X_1 + Z_2, \quad Z_2 \sim \mathcal{N}(0, N_2)
\]
\[
Y_3 = X_1 + X_2 + Z_3, \quad Z_3 \sim \mathcal{N}(0, N_3)
\]
\[
Y_4 = X_1 + X_2 + X_3 + Z_4, \quad Z_4 \sim \mathcal{N}(0, N_4)
\]

**NESTED LATTICE CODES** can mimic the regular encoding / sliding window decoding DF rate
• Unique decoding

• Intersection 2 lists

• Intersection 3 lists
Lists independent by independent mappings

Decoding

Block b-1

\[ Y_2 = X_1 + X_3 + Z_2 \]

\[ Y_3 = X_1 + X_2 + Z_3 \]

\[ Y_4 = X_1 + X_2 + X_3 + Z_4 \]

Block b

\[ x'_1 \leftrightarrow \Lambda_1 \subseteq \Lambda_{s(1-3)} \subseteq \Lambda_{s(1-4)} \subseteq \Lambda_{c1} \]

\[ x'_2 \leftrightarrow \Lambda_2 \subseteq \Lambda_{s(2-3)} \subseteq \Lambda_{s(2-4)} \subseteq \Lambda_{c2} \]

\[ x'_3 \leftrightarrow \Lambda_3 \subseteq \Lambda_{s(3-4)} \subseteq \Lambda_{c3} \]
Outline - enabling cooperation via lattices

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  - Lattices for multiple-access relay channel

- Lattices achieve CF rate for AWGN relay

(as known random)
Two-way relay channel (with direct links)
Two-way relay channel (with direct links)

\begin{align*}
Y_R &= X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R) \\
Y_1 &= X_1 + X_2 + X_R + Z_1 \quad Y_2 = X_1 + X_2 + X_R + Z_2 \\
Z_1 &\sim \mathcal{N}(0, N_1) \quad Z_2 \sim \mathcal{N}(0, N_2)
\end{align*}

- we derive a new achievable rate region using nested lattices, with direct link
- this region attains constant gaps for certain degraded channels
Theorem: For the two-way relay channel with direct links, we may achieve:

\[ R_1 \leq \min \left( \left[ \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right) \right]^+, \, \frac{1}{2} \log \left( 1 + \frac{P_1 + P_R}{N_2} \right) \right) \]

\[ R_2 \leq \min \left( \left[ \frac{1}{2} \log \left( \frac{P_2}{P_1 + P_2} + \frac{P_2}{N_R} \right) \right]^+, \, \frac{1}{2} \log \left( 1 + \frac{P_2 + P_R}{N_1} \right) \right) \]
\[
R_1 \leq \min \left( \left[ \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right) \right]^+, \left[ \frac{1}{2} \log \left( 1 + \frac{P_1 + P_R}{N_2} \right) \right]^+ \right)
\]
\[
R_2 \leq \min \left( \left[ \frac{1}{2} \log \left( \frac{P_2}{P_1 + P_2} + \frac{P_2}{N_R} \right) \right]^+, \left[ \frac{1}{2} \log \left( 1 + \frac{P_2 + P_R}{N_1} \right) \right]^+ \right)
\]

- eliminates “MAC”-like constraints at relay [Xie, CWIT, 2007]

\[
R_1 \leq \min \left( \frac{1}{2} \log \left( 1 + \frac{P_1}{N_R} \right), \frac{1}{2} \log \left( 1 + \frac{P_1 + P_R}{N_2} \right) \right)
\]
\[
R_2 \leq \min \left( \frac{1}{2} \log \left( 1 + \frac{P_2}{N_R} \right), \frac{1}{2} \log \left( 1 + \frac{P_2 + P_R}{N_1} \right) \right)
\]
\[
R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{N_R} \right)
\]

- combines direct and relayed information using lattice list decoder
\[ \hat{T} = (t_1 + t_2 - Q_2(t_2 + U_2)) \mod \Lambda_1 \]

- **Decoding sum**
  [Nam, Chung, Lee Trans. IT 2010]

- **BC with side-information**
  [Xie, CWIT 2007][Kramer, Shamai, ITW 2007] [Wu ISIT 2007] [Tuncel 2006]

- **List decoding**
  [Song, D 2010]

New achievable rate region

\[ S_{\mathcal{Y}_e, \Lambda_e} (\mathcal{Y}) = \{ \Lambda_e \cap (\mathcal{Y} + \mathcal{Y}_e) \} \]

\( \star = \mathcal{Y}' \)
intersect 2 lists

\[ \hat{T} \]

\[ L_{1-2}(w_1) \]
\[ L_{2-1}(w_2) \]

interact 2 lists

\[ X_R(s(\hat{T})) \]

\[ L_{R-1}(w_2) \]
\[ L_{R-2}(w_1) \]
**Encoding**

**Block b**

\[ X_1(w_{1b}) = (t_1(w_{1b}) - U_1) \mod \Lambda_1 \]

\[ X_2(w_{2b}) = (t_2(w_{2b}) - U_2) \mod \Lambda_2 \]

\[ X_R(s(\hat{T}(b - 1))) \]

**Decoding**

**Block b-1**

\[ \hat{T}(b - 1) = (t_1(w_{1(b-1)}) + t_2(w_{2(b-1)}) - Q_2(t_2(w_{2(b-1)}) + U_2)) \mod \Lambda_1 \]

**Block b**

\[ \hat{T}(b) = (t_1(w_{1b}) + t_2(w_{2b}) - Q_2(t_2(w_{2b}) + U_2)) \mod \Lambda_1 \]

**Intersect**

\[ L_{R-1}(w_{2b-1}) \]

\[ L_{R-2}(w_{1b-1}) \]

subtracts off decoded \( w_{2b-1} \)

\[ L_{2-1}(w_{2b-1}) \]

\[ L_{1-2}(w_{1b-1}) \]

subtracts off decoded \( w_{1b-1} \)

\[ L_{2-1}(w_{2b}) \]

\[ L_{1-2}(w_{1b}) \]
Outline of achievability scheme

- Relay node: \( Y_R = X_1 + X_2 + Z_R \), decodes \( \hat{T} \):

\[
R_1 < \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right) \\
R_2 < \frac{1}{2} \log \left( \frac{P_2}{P_1 + P_2} + \frac{P_2}{N_R} \right)
\]

- Node 2: \( Y_2 = X_1 + X_R + Z_2 \), decodes \( \hat{w}_1 \):

\[
R_1 < I(X_R; Y_2|X_2) + C(P_1/N_2) \\
= \frac{1}{2} \log \left( 1 + \frac{P_R}{P_1 + N_2} \right) + \frac{1}{2} \log \left( 1 + \frac{P_1}{N_2} \right) \\
= \frac{1}{2} \log \left( 1 + \frac{P_R + P_1}{N_2} \right).
\]

- Analogous for node 1
Outline - enabling cooperation via lattices

- Lattice notation
- Lattice list decoder

Single source DF applications:
- Lattices achieve DF rate for AWGN relay
- Lattices achieve DF rate for AWGN multi-relay

Multi-source DF applications:
- Lattices for two-way relay channel with direct links
- Lattices for multiple-access relay channel

Same rates as random
Lattices for the multiple-access relay channel

$Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R)$

$Y_D = X_1 + X_2 + X_R + Z_D, \quad Z_D \sim \mathcal{N}(0, N_D)$

Key idea: decode+forward sum at the relay
Lattices for the multiple-access relay channel

**Theorem:** The following rates are achievable for the AWGN multiple access relay channel:

\[ R_1 < \min \left( \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2 + \frac{P_1}{N_R}} \right) \right), \quad \frac{1}{2} \log \left( 1 + \frac{P_1 + P_R}{N_D} \right) \]

\[ R_2 < \min \left( \frac{1}{2} \log \left( \frac{P_2}{P_1 + P_2 + \frac{P_2}{N_R}} \right) \right), \quad \frac{1}{2} \log \left( 1 + \frac{P_2 + P_R}{N_D} \right) \]

\[ R_1 + R_2 < \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + P_R}{N_D} \right). \]

**Key idea:** decode+forward sum at the relay!
Decoding, order 1 then 2

$Y_D = X_1 + X_2 + X_R + Z_D$

switch order, time share

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Encoding

**Block b**

\[ X_1(w_{1b}) = (t_1(w_{1b}) - U_1) \mod \Lambda_1 \]

\[ X_2(w_{2b}) = (t_2(w_{2b}) - U_2) \mod \Lambda_2 \]

\[ X_R(s(\hat{T}(b - 1))) \]

Decoding, order 1 then 2

**Block b-1**

\[ \hat{T}(b - 1) = (t_1(w_{1(b-1)}) + t_2(w_{2(b-1)}) - Q_2(t_2(w_{2(b-1)}) + U_2)) \mod \Lambda_1 \]

**Block b**

\[ \hat{T}(b) = (t_1(w_{1(b)}) + t_2(w_{2(b)}) - Q_2(t_2(w_{2(b)}) + U_2)) \mod \Lambda_1 \]

\[ Y_R = X_1 + X_2 + Z_R \]

\[ Y_D = X_1 + X_2 + X_R(s(\hat{T})) + Z_D \]

\[ L_{R-D}(w_{1(b-1)}) \]

\[ L_{R-D}(w_{2(b-1)}) \]

\[ L_{2-D}(w_{2(b-1)}) \]

\[ L_{2-D}(w_{2b}) \]

Intersect

\[ L_{R-D}(w_{1(b-1)}) := \{ w_{1(b-1)} : (x_R(s(T(b - 1))), X_2(w_{2b}), Y_D(b)) \in A^N \} \]

\[ L_{R-D}(w_{2(b-1)}) := \{ w_{2(b-1)} : (x_R(s(T(b - 1))), X_1(w_{1b}), Y_D(b)) \in A^N \} \]
Outline of achievability scheme

- Relay node: \( Y_R = X_1 + X_2 + Z_R \), decodes \( \hat{T} \):

\[
R_1 \leq \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right)
\]

\[
R_2 \leq \frac{1}{2} \log \left( \frac{P_2}{P_1 + P_2} + \frac{P_2}{N_R} \right)
\]

- Node D decode \( w_{1b} \) if \( R_1 < \frac{1}{2} \log \left( 1 + \frac{P_1}{P_2 + P_R + N_D} \right) \)

- Node D decode \( w_{2(b-1)} \) if

\[
R_2 < I(X_R : Y_2 | X_1) + C \left( \frac{P_2}{N_D} \right)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{P_R}{P_2 + N_D} \right) + \frac{1}{2} \log \left( 1 + \frac{P_2}{N_D} \right)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{P_2 + P_R}{N_D} \right)
\]

- Order \( w_{1b} \) then \( w_{2b} \)

\[\text{Reverse order and time-share}\]

- subtract \( w_{1b}, w_{1(b-1)}, w_{2(b-1)} \) and decode list \( L_{2-D}(w_{2b}) \) of size \( 2^n(R_2 - C(P_2/N_D)) \)
Outline - enabling cooperation via lattices

• Lattice notation

• Lattice list decoder

• Single source DF applications:
  • Lattices achieve DF rate for AWGN relay
  • Lattices achieve DF rate for AWGN multi-relay
  \[
  \text{Same rates as random}
  \]

• Multi-source DF applications:
  • Lattices for two-way relay channel with direct links
  • Lattices for multiple-access relay channel
  \[
  \text{Sometimes better rates than random}
  \]

• Lattices achieve CF rate for AWGN relay
Why sometimes better?

Structured

\[ R_1 \leq \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right) \]
\[ R_2 \leq \frac{1}{2} \log \left( \frac{P_2}{P_1 + P_2} + \frac{P_2}{N_R} \right) \]

- no coherent gain (yet) when decode sum

Random

\[ R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{N_R} \right) \]
\[ R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_2}{N_R} \right) \]
\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{N_R} \right) \]

- coherent gains

(Tx needs to know exact relay message, does not if decode sum)
Outline - enabling cooperation via lattices

- Lattice notation
- Lattice list decoder

Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay

Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel

Lattices achieve CF rate for AWGN relay

\(\text{Same rates as random}\)

\(\text{Sometimes better rates than random}\)
Compress and forward (CF)

- DF limited by need to decode at relay
- CF is NOT limited in this fashion

\[ Y_R \rightarrow \hat{Y}_R \]

\[ \hat{Y}_R \rightarrow X_R \text{ Wyner-Ziv} \]

direct link side-information

combine with direct link
The \((X+Z_1, X+Z_2)\) Wyner-Ziv problem

- Gaussian Wyner-Ziv \((X + Z_1, X + Z_2)\)

- in [Zamir, Shamai, Erez, 2002] demonstrated a lattice scheme for Gaussian \((X + Z, X)\) Wyner-Ziv which is fully general

- demonstrate \((X + Z_1, X + Z_2)\) for completeness
The $(X+Z_1, X+Z_2)$ Wyner-Ziv problem

**Theorem.** The following rate-distortion function for the lossy compression of the source $X + Z_1$ subject to the reconstruction side-information $X + Z_2$ and squared error distortion metric may be achieved using lattice codes:

$$R(D) = \frac{1}{2} \log \left( \frac{\sigma^2_{X+Z_1|X+Z_2}}{D} \right), \quad 0 \leq D \leq \sigma^2_{X+Z_1|X+Z_2}$$

$$= \frac{1}{2} \log \left( \frac{N_1 + \frac{PN_2}{P+N_2}}{D} \right), \quad 0 \leq D \leq N_1 + \frac{PN_2}{P+N_2},$$

and 0 otherwise.

\[
\begin{align*}
Y &= X + Z_1 \\
\alpha_1 &\quad Q_q(\cdot) \quad \text{mod } \Lambda \\
\oplus &\quad X_1 \\
\mathsf{Encoding (function } f(\cdot)) &\quad \mathsf{Decoding (function } g(\cdot, \cdot))
\end{align*}
\]
A Lattice CF scheme

**Theorem.** For the three user Gaussian relay channel described by the input/output equations $Y_R = X_1 + N_R$ at the relay’s receiver and $Y_2 = X_1 + X_R + N_2$ at the destination, with corresponding input and noise powers $P_1, P_R, N_R, N_2$, the following rate may be achieved using lattice codes in a lattice Compress-and-Forward fashion:

$$R < \frac{1}{2} \log \left( 1 + \frac{P_1}{N_2} + \frac{P_1 P_R}{P_1 N_R + P_1 N_2 + P_R N_R + N_R N_2} \right).$$

**same as that achieved by Gaussian codes in the CF scheme of [Cover, El Gamal, 1979]**
**Encoding**

*Block b*

\[ x_1(w_b) \]

\[ x_R(w_{b-1}), \text{encoding of index } i \text{ of compression of } Y_R(w_{b-1}) = X_1(w_{b-1}) + Z_2 \]

**Decoding**

*Block b-1*

\[ Y'_2(w_{b-1}) = X_1(w_{b-1}) + Z_2 \]

*Wyner-Ziv*

- compression index \( i \) from \( Y_2 = X_1 + X_r + Z_2 \)
- \( Y'_2(w_b) = Y_2(w_b) - X_R(w_{b-1}) = X_1(w_b) + Z_2 \)
- \( X_R(w_{b-1}) \) to reconstruct \( \hat{Y}_R(w_{b-1}) \)

*Block b*

- coherently combine \( Y'_2(w_{b-1}) \) and \( \hat{Y}_R(w_{b-1}) \) to decode \( w_{b-1} \)

\[
\begin{align*}
\text{rate } R & \quad w \leftrightarrow t_1, \, \Lambda_1 \subseteq \Lambda_{c1}, \sigma^2(\Lambda_1) = P_1 \\
\text{compression } \Lambda & \subseteq \Lambda_q, \quad \text{rate } \hat{R}, \quad \sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D \\
i & \leftrightarrow t_R, \, \Lambda_R \subseteq \Lambda_{cR}, \quad \text{rate } R', \quad \sigma^2(\Lambda_R) = P_R
\end{align*}
\]
compression $\Lambda \subseteq \Lambda_q$ rate $\hat{R}$ 

$$\sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D$$

$i \leftrightarrow t_R$, $\Lambda_R \subseteq \Lambda_{cR}$ rate $R'$ 

$$\sigma^2(\Lambda_R) = P_R$$

rate $R$ $w \leftrightarrow t_1$, $\Lambda_1 \subseteq \Lambda_{c1}$ 

$$\sigma^2(\Lambda_1) = P_1$$

- decodes $i$ from $Y_2 = X_1 + X_R + Z_2$ as long as $R' < \frac{1}{2} \log_2 \left( 1 + \frac{P_R}{P_1 + N_2} \right)$

- source coding < channel coding rate: $\frac{1}{2} \log \left( 1 + \frac{N_R + \frac{P_1 N_2}{P_1 + N_2}}{D} \right) < \frac{1}{2} \log \left( 1 + \frac{P_R}{P_1 + N_2} \right)$

- use $Y'_2 = Y_2 - X_R$ from previous block and $X_R$ from current block to reconstruct $\hat{Y}_R$ as in $(X + Z_R, X + Z_2)$ Wyner-Ziv

- coherently combine $Y'_2$ and $\hat{Y}_R$ to decode $w$, as long as $R < \frac{1}{2} \log \left( 1 + \frac{P_1}{N_2} + \frac{P_1}{N_R + D} \right)$

- note: pick $\alpha_1 = 1$ rather than source coding MMSE to render compression noise independent of all else
Enabling lattice "Cooperation"

Lattice list decoder

Lattices achieve DF rate

Intersect 2 lists

Lattices in multi-source networks

Lattices achieve CF rate

Lattices good source and channel codes, special Wyner-Ziv
Conclusion

• can random codes be replaced by structured codes in Gaussian networks?

• capacity of relay channel? can structured codes help?

• list decoding - can help in explaining transmission above capacity?
Questions?

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