Scaling Laws of Cognitive Networks

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***Abstract*—Opportunistic secondary spectrum usage has the potential to dramatically increase spectral efficiency and rates of a network of secondary cognitive users. In this work we consider a *cognitive network:* n pairs of cognitive transmitter and receiver wish to communicate simultaneously in the presence of a single primary transmitter-receiver link. We assume each cognitive transmitter-receiver pair communicates in a realistic single-hop fashion, as cognitive links are likely to be highly localized in space. We first show that under an outage constraint on the primary link's capacity, provided that the density of the cognitive users is constant, the sum-rate of the n cognitive links scales linearly with n as $n \to \infty$. This scaling is in contrast to the sum-rate scaling of $\sqrt{n}$ seen in multi-hop ad-hoc networks. We then explore the optimal radius of the *primary exclusive region*: the region in which no secondary cognitive users may transmit, such that the outage constraint on the primary user is satisfied. We obtain bounds that help the design of this primary exclusive region, outside of which cognitive radios may freely transmit.**

I. Introduction

Currently, spectrum access is granted to entities in an exclusive *primary license* fashion: licenses in certain bands are auctioned off to the private sector, while others are dedicated for use by officials in the public sector. Unlike the unlicensed bands (such as the popular Wifi band), measurements indicate that many of the licensed bands remain unoccupied for large chunks of time, space, and frequency. The Federal Communications Commission (FCC) is trying to remedy this imbalance through the introduction of *secondary spectrum licensing*, whereby primary license holders may easily grant non-licensed holders opportunistic usage of the spectrum. Naturally, secondary users will only be granted access provided the primary users suffer only an acceptable amount of degradation in performance (if any at all).

Consider for example a TV station broadcasting in a now-exclusive, licensed band. This band is wasted in geographic locations barely covered by the TV signal. This prompts questions such as: can we allow other devices to transmit in the same band as the TV, provided their interference to any TV receiver is at "an acceptable level"? If so, what is the minimum distance from the TV station at which these devices can start transmitting? What are the maximum rates that these devices can achieve by transmitting in the TV band?

We formulate this problem from an information theoretic viewpoint as a cognitive network. In a cognitive network, there is a primary user (e.g. the TV station) and multiple secondary (cognitive) users. We define the "acceptable interference level" to be a threshold on the probability that the received signal (or rate) of the primary user is below a certain level, provided that the primary receiver (Rx) is within a radius of interest from its transmitter (Tx). This is analogous to the concept of outage capacity. The radius of interest specifies the primary exclusive region, outside of which, the cognitive users can communicate among themselves. We consider the scenario in which the cognitive transmitters are uniformly distributed such that their density is a constant. We further assume that each cognitive transmitter communicates with a receiver within a bounded distance, independent of the network size. The cognitive communication therefore occurs in a single hop. This assumption appears reasonable for secondary spectrum usage, which is opportunistic in nature and hence is often a local, single-hop transmission.

This work is closely related to results on ad-hoc network capacity scaling laws. Initiated by the work of Gupta and Kumar [1], this area of research has been actively pursued under a variety of wireless channel models and communication protocol assumption [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. These papers usually assume n pairs of devices, thrown at random in a plane, wishing to communicate. Each transmitter has a single receiver. The question they seek to answer is how the total network capacity (or sum rate, or throughput) scales as the number of communication pairs $n \to \infty$. This is accomplished by either letting the density of nodes stay fixed and the area increase with n (extended network) or by fixing the network area and letting the density increase with n (dense network). The results in the literature can be roughly grouped into two types: when nodes in the ad-hoc network cannot cooperate (forwarding a message in a multi-hop transmission is not considered to be cooperation), it has been shown that the per user network capacity decreases as $1/\sqrt{n}$ as $n \to \infty$ [1], [2], [5]. This is essentially thought of as a negative result, and can be viewed as a consequence of the unmitigated interference experienced. In contrast, when nodes are able to cooperate, it can be shown that the per user capacity remains constant [11]. All of these scaling results naturally depend on the path loss parameter, which affects both the desired signal transmission and the interference experienced from other transmitting users.

Our network setup is equivalent to an extended and interference-limited network (no cooperation allowed). Under the single-hop assumption and bounded cognitive Tx-Rx distance, we show that the total sum-rate of the cognitive network increases *linearly* in the number of cognitive users n. Equivalently, in the limit as the number of cognitive users tends to infinity, the *per user capacity* remains constant. This result is in sharp contrast to the per-user capacity decreasing at $1/\sqrt{n}$ in a multi-hop ad-hoc network [1]. Here, single-hop transmission for the cognitive users is the key difference that enables the linear-scaling sum rate.

This linear scaling law also implies that the average total interference from all cognitive transmitters to the primary user's receiver remains bounded by a constant irrespective of the number of cognitive users. Based on this interference bound, we provide an upper bound on the radius of the primary exclusive region that satisfies the outage constraint on the primary user's rate.

This paper is structured as follows. In Section II, we formulate the problem: we consider a single primary user at the center of a network wishing to communicate with a primary receiver located within the primary exclusive region of radius R_0. In the same plane outside this region, we throw n cognitive users, each of which wishes to transmit to its own cognitive receiver within a fixed radius away. In Section III, we obtain lower and upper bounds on the total sum-rate of the n cognitive users as $n \to \infty$ and establish the scaling law. In Section IV, we examine the outage constraint on the primary user's rate in terms of cognitive node placement: that is, we explicitly derive the *exclusive region* radius R_0 around the primary, in which the primary user have the *exclusive* right to transmit and no cognitive users may do so. In Section V we make our conclusions.

II. Problem Formulation

We are interested in the general question of how much total data rate can be exchanged among the cognitive users, provided that the probability of the resulting outage on the primary user is below a certain level. Since the users are not allowed to cooperate, the setup is similar to an interference channel. We will first discuss the network model and characterize the signal of and the interference at each user.

A. Network model

We introduce our network model in Fig. 1. We assume that all users' transmitters and receivers are distributed on a plane. Let Tx^0 and Rx^0 denote the primary transmitter and receiver, while Tx^i and Rx^i are pairs of secondary transmitters and receivers, respectively, $i = 1, 2, \cdots, n$. The primary transmitter is located at the center of the primary exclusive region with radius R_0, while all the cognitive transmitters and receivers are distributed in a ring outside this exclusive region with an outer radius R. We assume that the cognitive transmitters are located randomly and uniformly in the ring. Each cognitive receiver, however, is within a $D_{\max}$ distance from its transmitter. Furthermore, the cognitive user density is constant at λ users per unit area. The outer radius R therefore grows as the number of cognitive users increases.

The notation is summarized in Table I.

B. Signal and interference characteristics

The received signal at Rx^0 is denoted by y_0, while that at Rx^i is denoted by y_i. These relate to the signals x_0 transmitted by the primary Tx^0 and x_i by the cognitive Tx^i as

$$y_0 = h_0 x_0 + \sum_{i=1}^{n} h_i x_i + n_0 \quad (1)$$

$$y_i = h_{ii} x_i + g_i x_0 + \sum_{j \neq i} h_{ji} x_j + n_i. \quad (2)$$

From this signal model, several **assumptions** can be made:

- Different users' signals are statistically independent. The primary user signal is constrained by a power P_0, and each cognitive user by P.
- The number of cognitive users is large (n large). Because these users are independent and power-constrained, their interference to the primary user in (1) is (approximately) Gaussian. The Gaussian noise and interference means that the optimal transmit signal x_0 in (1) is Gaussian [12].
- Similarly, the total interference of the cognitive users on

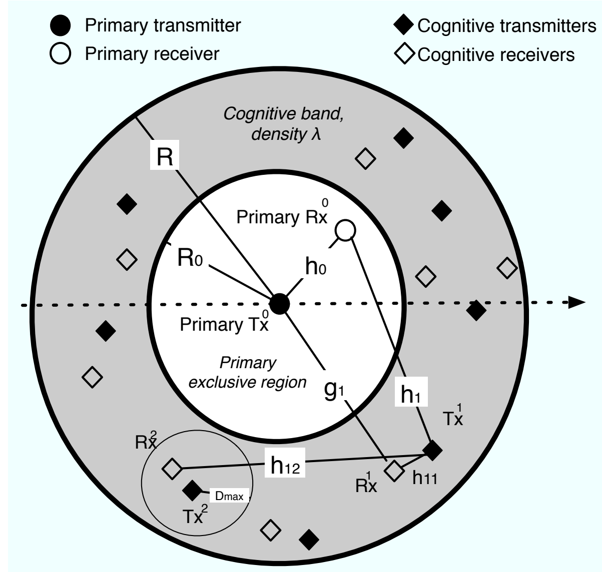

Fig. 1. A cognitive network: a single primary transmitter Tx^0 is placed at the origin and wishes to transmit to its primary receiver Rx^0 in the circle of radius R_0 (the *primary exclusive region*. The n cognitive nodes are randomly placed with uniform density λ in the shaded *cognitive band*. The cognitive transmitter Tx^i wishes to transmit to a single cognitive receiver Rx^i which lies within a distance $< D_{max}$ away. The cognitive transmissions must satisfy a primary outage constraint.

Primary transmitter and receiver	Tx^0, Rx^0
Cognitive user ith transmitter and receiver	Tx^i, Rx^i
Primary exclusive region radius	R_0
Outer radius for cognitive transmission	R
Channel from Tx^0 to Rx^0	h_0
Channel from Tx^0 to Rx^i	g_i
Channel from Tx^i to Rx^0	h_i
Channel from Tx^i to Rx^j	h_{ij}
Number of cognitive users	n
Maximum cognitive Tx^i-Rx^i distance	$D_{\max}$
Cognitive user density	λ

TABLE I
VARIABLE NAMES AND DEFINITIONS

each other in (2) is (approximately) Gaussian. Because the optimal x_0 is Gaussian, the total noise and interference for each cognitive user in (2) is also Gaussian.
- The cognitive users have no knowledge of each other's signal and hence treat their interference as noise. The optimal transmit signal x_i for each cognitive user therefore is also Gaussian.
- All the signals and noises have zero-mean.

C. Channel model

We consider a path-loss only model for the wireless channel. Given a distance d between the transmitter and the receiver, the channel is therefore given as

$$h = \frac{A}{d^{\alpha/2}} \tag{3}$$

where A is a frequency-dependent constant and α is the power path loss. We consider $\alpha > 2$ which is typical in practical scenarios.

We are interested in two measures: the sum rate of all cognitive users and the optimal radius of the primary exclusive region. The cognitive network sum rate is given as

$$C_n = \sum_{i=1}^{n} I(x_i; y_i) = \sum_{i=1}^{n} C_i,$$

where $C_i = I(x_i; y_i)$ denotes the mutual information between random variables x_i, the signal transmitted by the cognitive user Tx^i, and y_i, the signal received by the cognitive receiver Rx^i. This quantity C_i corresponds to an achievable rate of the point to point channel between Tx^i and Rx^i. Here n again denotes the total number of cognitive users. The radius R_0 of the primary exclusive region is determined by the outage constraint on the primary user given as

$$\Pr\left[I(x_0; y_0) \le C_0\right] \le \beta$$

where C_0 and β are pre-chosen constants. This constraint guarantees the primary user a rate of at least C_0 for all but β fraction of the time.

Assume that each cognitive user transmits with the same power P, and the primary user transmits with power P_0. Denote I_i $(i = 0, \ldots, n)$ as the total interference power from the cognitive transmitters to user i, then

$$I_0 = \sum_{i=1}^{n} P|h_i|^2 \tag{4}$$

$$I_i = \sum_{j \ne i} P|h_{ji}|^2 \tag{5}$$

By the signal models in (1) and (2) and the assumptions in part II-B, the rate of each cognitive user can be written as

$$C_i = \log\left(1 + \frac{P|h_{ii}|^2}{P_0|g_i|^2 + \sigma_{ni}^2 + I_i}\right), \quad i = 1 \ldots n. \tag{6}$$

The outage constraint can now be written as

$$\Pr\left[\log\left(1 + \frac{P_0|h_0|^2}{\sigma_{n0}^2 + I_0}\right) \le C_0\right] \le \beta. \tag{7}$$

In this work we assume the channel gains depend only on the distance between transmitters and receivers as in (3), and do not suffer from fading or shadowing. Thus, all randomness is a result of the random distribution of the cognitive nodes in the cognitive band of Fig. 1.

III. The Scaling Law of a Cognitive Network

In this section, we study the sum capacity of the cognitive network. In particular, we examine its scaling law as the number of cognitive users n increases to infinity. Since we consider only a single primary transmitter with fixed power P_0 and minimum distance R_0 from any cognitive receiver, the interference from this primary user is bounded by a constant and therefore has no impact in the asymptotic rate analysis. We can therefore treat the interference from the primary user as an additive noise term.

A. Lower bound on the network sum capacity

To derive a lower bound on the network sum capacity, we study an upper bound on the interference to a cognitive receiver. An upper bound is obtained by filling the primary exclusive region with cognitive users. Since the primary exclusive region is fixed and the cognitive user density is constant, the average number of cognitive users filled in this region is a constant. Thus this filling does not affect the asymptotic interference as $n \to \infty$.

Now consider a uniform network of n cognitive users. The worst case interference then is to the user with the receiver at the center of the network. From the considered receiver (WLOG assumed to be Rx^1), draw a circle of radius R_c that covers all other cognitive transmitters. With constant user density of λ users per unit area, then R_c satisfies $\lambda \pi R_c^2 = n$. In other words, R_c^2 is of order n.

To see that this case is the worst interference, consider another cognitive receiver (Rx^2) that is not at the center of the network. Again draw a circle of radius R_c centered at Rx^2. Since this receiver is not at the center of the network, the circle will not cover all cognitive transmitters. The interference to Rx^2 is then increased by moving all the transmitters from outside this new circle (area A in Fig. 2) to inside the circle (area B in Fig. 2), resulting in the same interference as to Rx^1.

In deriving a lower bound on network sum capacity, we further assume that any interfering cognitive transmitter must be at least a distance ϵ away from the interfered receiver for some $\epsilon > 0$. This assumption is practically reasonable.

Consider an interfering cognitive transmitter located randomly within the circle of radius R_c from the considered receiver. With uniform distribution, the distance r between this interfering transmitter and the considered receiver has the density

$$f_r(r) = \frac{2r}{R_c^2 - \epsilon^2}, \quad \epsilon \le r \le R_c .$$

The *average interference* from this transmitter to the consid-

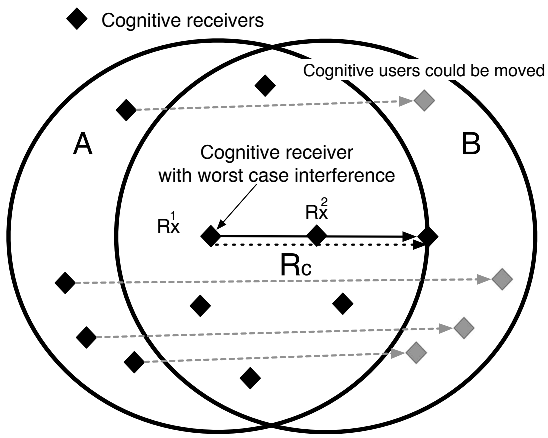

Fig. 2. Worst-case interference to a cognitive receiver.

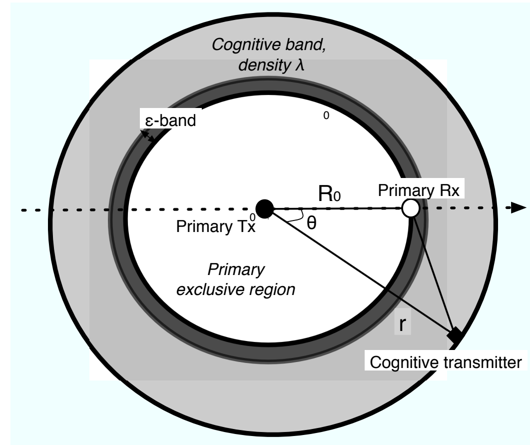

Fig. 3. Worst-case interference to a primary receiver: the receiver is on the boundary of the primary exclusive region of radius R_0. We seek to find R_0 such that the outage constraint on the primary channel is met.

ered receiver therefore is

$$I_{\text{avg},1} = \int_{\epsilon}^{R_c} \frac{2rP}{(R_c^2-\epsilon^2)r^{\alpha}} dr \tag{8}$$

$$= \frac{2P}{(R_c^2-\epsilon^2)(\alpha-1)} \left(\frac{1}{\epsilon^{\alpha-2}} - \frac{1}{R_c^{\alpha-2}} \right). \tag{9}$$

The average total interference from all other cognitive transmitters to the considered receiver then becomes

$$I_{\text{avg},n} = n I_{\text{avg},1} \; .$$

But $\lambda\pi(R_c^2-\epsilon^2) = n$, thus

$$I_{\text{avg},n} = \frac{2\pi\lambda P}{(\alpha-1)} \left(\frac{1}{\epsilon^{\alpha-2}} - \frac{1}{R_c^{\alpha-2}} \right). \tag{10}$$

As $n \to \infty$, provided that $\alpha > 2$, this average interference to the cognitive receiver at the center approaches a constant as

$$I_{\text{avg},n} \overset{n\to\infty}{\longrightarrow} \frac{2\pi\lambda(\alpha-1)}{\epsilon^{\alpha-2}} \triangleq I_{\infty}. \tag{11}$$

For any cognitive receiver, its average interference is upper-bounded by $I_{\text{avg},n}$, that is

$$E[I_i] \leq I_{\text{avg},n} \; . \tag{12}$$

Now consider the rate of the ith cognitive user given in (6). Since the distance between a cognitive transmitter and its intended receiver is bounded by $D_{\max}$, we have $|h_{ii}|^2 \geq 1/D_{\max}^{\alpha}$. Furthermore, $|g_i|^2 \leq 1/R_0^{\alpha}$. Denote the minimum received power as $P_{r,\min}/D_{\max}^{\alpha}$ and the maximum noise and primary user interference as $\sigma_{0,\max}^2 \sigma_n^2 + P_0/R_0^{\alpha}$, then

$$C_i \geq \log\left(1 + \frac{P_{r,\min}}{\sigma_{0,\max}^2 + I_i}\right). \tag{13}$$

Noting that $\log(1+a/x)$ is convex in x for $a > 0$, by Jensen's inequality, we have

$$E \log\left(1+\frac{a}{X}\right) \geq \log\left(1+\frac{a}{EX}\right).$$

Thus the average rate of each cognitive user satisfies

$$\begin{aligned} E[C_i] \quad &\geq \quad \log\left(1 + \frac{P_{r,\min}}{\sigma_{0,\max}^2 + E[I_i]}\right) \\ &= \quad \log\left(1 + \frac{P_{r,\min}}{\sigma_{0,\max}^2 + I_{\text{avg},n}}\right). \end{aligned} \tag{14}$$

As $n \to \infty$, the lower bound approaches a constant as

$$E[C_i] \geq \log\left(1 + \frac{P_{r,\min}}{\sigma_{0,\max}^2 + I_{\infty}}\right) \triangleq \bar{C}_1. \tag{15}$$

Thus the average per-user rate of a cognitive network remains at least a constant as the number of users increases.

B. Upper bound on the network sum capacity

A trivial upper-bound can be obtained by removing the interference from all other cognitive users. Assuming that the capacity of a single cognitive user under noise alone is bounded by a constant, then the total network capacity grows at most linearly with the number of users.

C. Linear scaling law of the cognitive network sum capacity

From the above lower and upper bounds, we conclude that the sum capacity of the cognitive network grows linearly in the number of users

$$E[C_n] = nK\bar{C}_1$$

for some constant K, where $\bar{C}_1$ defined in (15) is the achievable average rate of a single cognitive user under constant noise and interference power.

IV. The Primary Exclusive Region

To study the primary exclusive region, we consider the worst case when the primary receiver is at the edge of this region, on the circle of radius R_0, as shown in Fig.3. The outage constraint must also hold in this (worst) case, and we find a bound on R_0 that will ensure this.

Assume that the primary receiver is at a point on the boundary of the exclusive region. Consider interference at this point from a cognitive transmitter at radius r and angle θ. The distance $d(r,\theta)$ (the distance depends on r and θ) between this interfering transmitter and the primary receiver satisfies

$$d(r,\theta)^2 = R_0^2 + r^2 - 2R_0 r\cos\theta\ .$$

For uniformly distributed cognitive users, θ is uniform in $[0, 2\pi]$, and r has the density $f_r(r) = 2r/(R^2 - (R_0+\epsilon)^2)$. We assume that the cognitive transmitters must be placed minimally at a radius $R_0 + \epsilon$, thus cannot be placed in the transmission-free ϵ-band in Fig.3. This assumption is valid in all scenarios where the cognitive transmitter is forbidden to be placed in exactly the same location as the primary receiver. The expected interference plus noise power experienced by the primary receiver Rx^0 from all $n = \lambda\pi(R^2 - (R_0+\epsilon)^2)$ cognitive users is then given as

$$\begin{aligned} E[I_0] &= \int_{R_0+\epsilon}^{R}\int_0^{2\pi} \frac{P}{d(r,\theta)^2} f_r(r) f_\theta(\theta)\ dr\ d\theta \\ &= \int_{R_0+\epsilon}^{R}\int_0^{2\pi} \frac{2rP\ dr\ d\theta}{2\pi(R_0^2 + r^2 - 2R_0 r\cos\theta)^{\alpha/2}} \end{aligned} \tag{16}$$

Applying the bounds $-1 \le \cos(\theta) \le 1$ to (16), we can upper and lower bound this expected interference as (17)–(19). When we let the number of users $n \to \infty$, or equivalently, $R \to \infty$, we obtain the bounds on the total interference seen by the worst case primary receiver $E[I_0]_\infty$ in (20), which demonstrate explicitly the dependence on the power loss parameter α. We note that these bounds on the primary user interference are valid for any $\alpha > 2$.

When $\alpha/2$ is an integer, we may evaluate the integral for the exact interference using complex contour integration techniques. As an example, we work out the result explicitly for $\alpha = 4$ in the Appendix, and state the result in (21) below.

$$E[I_0] = \lambda\pi P\left[-\frac{R^2}{(R^2-R_0^2)^2} + \frac{(R_0+\epsilon)^2}{\epsilon(2R_0+\epsilon)^2}\right]. \tag{21}$$

Bounds on the radius R_0 of the primary exclusive region may then be obtained by applying Markov's inequality, for given C_0, β, to the primary outage constraint (7) as

$$\begin{aligned} & \Pr\left[\log_2\left(1 + \frac{P_0/R_0^2}{I_0 + \sigma_{n0}^2}\right) \le C_0\right] \\ = \ & \Pr\left[I_0 \ge \frac{P_0/R_0^2}{(2^{C_0}-1)} - \sigma_{n0}^2\right] \\ \le \ & \frac{E[I_0]}{\frac{P_0/R_0^2}{(2^{C_0}-1)} - \sigma_{n0}^2} \\ = \ & \frac{\lambda\pi P\left[-\frac{R^2}{(R^2-R_0^2)^2} + \frac{(R_0+\epsilon)^2}{\epsilon(2R_0+\epsilon)^2}\right]}{\frac{P_0/R_0^2}{(2^{C_0}-1)} - \sigma_{n0}^2}. \end{aligned}$$

Letting $R \to \infty$, and bounding this by β (the outage level), we obtain the implicit equation (22) for all exclusive region radii R_0 such that (7) holds:

$$\frac{(R_0+\epsilon)^2}{\epsilon(2R_0+\epsilon)^2} \le \frac{\beta}{\lambda\pi P}\left(\frac{P_0/R_0^2}{2^{C_0}-1} - \sigma_{n0}^2\right) \tag{22}$$

Given the system parameters P_0, β, C_0, one can use (22) to design the exclusive region radius R_0 and the band ϵ to meet the desired outage constraint.

V. Conclusion

As secondary spectrum usage is rapidly approaching, it is important to study the potential of cognitive radios and transmission from a *network* perspective. In this work, we have determined the sum-rate scaling of a network of one-hop cognitive transmitter-receiver pairs which simultaneously communicate, while probabilistically guaranteeing the primary user link a minimum rate. When cognitive transmitters simultaneously transmit to nearby receivers, we show that the sum-rate scaling scales linearly in the number of cognitive links n as $n \to \infty$. Our work assumes a single primary link with an outage constraint; it would be interesting to extend these results to the case of multiple primary links. In this work, we also derive bounds which allow one to design the primary exclusive region, in which no cognitive transmission may take place. If properly chosen, outside this region, uniformly distributed cognitive transmitters may freely transmit while not harming the primary user.

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$$\frac{1}{(r+R_0)^\alpha} = \int_0^{2\pi} \frac{d\theta}{2\pi(r+R_0)^\alpha} \quad \le \int_0^{2\pi} \frac{d\theta}{2\pi(R_0^2+r^2-2R_0 r\cos\theta)^{\alpha/2}} \le \quad \int_0^{2\pi} \frac{d\theta}{2\pi(r-R_0)^\alpha} = \frac{1}{(r-R_0)^\alpha} \tag{17}$$

$$\lambda\pi \int_{R_0+\epsilon}^{R} \frac{2rP\ dr}{(r+R_0)^\alpha} \le E[I_0] \le \lambda\pi \int_{R_0+\epsilon}^{R} \frac{2rP\ dr}{(r-R_0)^\alpha} \tag{18}$$

$$2\lambda\pi P\left[\frac{-\frac{R+R_0}{(\alpha-2)}-\frac{P_0}{(\alpha-1)}}{(R+R_0)^{\alpha-1}}+\frac{\frac{2R_0+\epsilon}{(\alpha-2)}-\frac{R_0}{(\alpha-1)}}{(2R_0+\epsilon)^{\alpha-1}}\right] \le E[I_0] \le 2\lambda\pi P\left[\frac{-\frac{R-R_0}{(\alpha-2)}-\frac{P_0}{(\alpha-1)}}{(R-R_0)^{\alpha-1}}+\frac{\frac{\epsilon}{(\alpha-2)}+\frac{R_0}{(\alpha-1)}}{\epsilon^{\alpha-1}}\right] \tag{19}$$

$$\Downarrow$$

$$2\lambda\pi P\left[\frac{\frac{2R_0+\epsilon}{(\alpha-2)}-\frac{R_0}{(\alpha-1)}}{(2R_0+\epsilon)^{\alpha-1}}\right] \le E[I_0]_\infty \le 2\lambda\pi P\left[\frac{\frac{\epsilon}{(\alpha-2)}+\frac{R_0}{(\alpha-1)}}{\epsilon^{\alpha-1}}\right] \tag{20}$$

Appendix

For $a > |b|$, from pg. 383 [13], we obtain

$$\int_0^{2\pi} \frac{dx}{(a+b\cos(x))^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

In the integral of interest (16) we have $a = R_0^2 + r^2$ and $b = -2R_0 r$, and so $R_0^2 + r^2 > 2R_0 r$ as needed. Thus, the expected interference from all cognitive users is given by (23).

$$\begin{aligned} E[I_0] &= \lambda\pi P \int_{R_0+\epsilon}^{R}\int_0^{2\pi} \frac{2r\ dr\ d\theta}{2\pi(R_0^2+r^2-2R_0 r\cos\theta)^2} \\ &= \lambda\pi P \int_{R_0+\epsilon}^{R^2} \frac{2r(r^2+R_0^2)}{(r^2-R_0^2)^3}\ dr \\ &= \lambda\pi P\left[-\frac{r^2+R_0^2}{2(r^2-R_0^2)^2}-\frac{1}{2(r^2-R_0^2)}\right]\Bigg|_{R_0+\epsilon}^{R} \\ &= \lambda\pi P\left[-\frac{R^2}{(R^2-R_0^2)^2}+\frac{(R_0+\epsilon)^2}{\epsilon(2R_0+\epsilon)^2}\right] \end{aligned} \tag{23}$$

Thus, if we let the number of users $n \to \infty$, or equivalently, as $R \to \infty$, the total interference experienced by the primary receiver when on the edge of the primary exclusive region approaches the constant

$$E[I_0]_\infty = \frac{\lambda\pi P(R_0+\epsilon)^2}{\epsilon(2R_0+\epsilon)^2}.$$