The Capacity Region of Gaussian Cognitive Radio Channels to within 1.87 bits

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***Abstract*—We make use of the deterministic high-SNR approximation of the Gaussian cognitive radio channel to gain insights in deriving inner and outer bounds for any SNR. We show that the derived bounds are at most 1.87 bits apart for any SNR.**

I. Introduction

Advancements in wireless technology have enabled the cooperation among devices, ensuring faster and more reliable communication. With smart and well-interconnected wireless transmitters, collaboration among entities is a relevant topic for modern communication systems. A well known and studied channel model inspired by the newfound abilities of cognitive radio technology and its potential impact on spectral efficiency in wireless networks is the *cognitive radio channel* [2]. Here the "primary" and the "secondary" transmitters each have a message destined to the primary and secondary receivers, respectively. Over the shared communication channel each transmitted message interferes with the other at the destination. The secondary transmitter has full a-priori knowledge of the primary message: this assumption is referred to as *cognition*. The capacity region of the cognitive radio channel, both for discrete memoryless as well as Gaussian noise channel remains unknown in general. For the Gaussian case, capacity is known in the weak interference [15], [3] and the very strong interference [4] regimes.

In the last couple of years an alternative to the difficult task of determining the capacity region of a multi user communication network has been suggested. Rather than proving an equality between inner and outer bounds, the authors [6] advocate a powerful new method for obtaining achievable rate regions that lie within a constant number of bits from capacity region outer bounds, thereby determining the capacity region to "within a constant number of bits". In a series of papers, and inspired by [7], Avestimehr, Diggavi and Tse introduced the linear deterministic approximation of wireless networks [8], [9], [12]. The deterministic model is often easier to analyze than the original Gaussian channel in that its capacity can be determined exactly. The insights gained from the analysis of the deterministic model are then used to guide the design of coding schemes for inner bounds and the choice of receiver side-information for outer bounds for the practically motivated Gaussian noise channel. Ideally, the bounds are then shown to lie within a constant gap from each other. Finite bit gaps between inner and outer bounds have been shown for channels whose capacity regions have been long standing open problems, such as Gaussian interference channels [10], [11], [17] and Gaussian relay channels [12].

In this work we use the same apprach. In our recent work [13] we introduced the high-SNR approximation model of the Gaussian cognitive channel and determined its capacity. In this paper we use the insights derived from [13] to determine a finite gap result for the Gaussian cognitive interference channel. The rest of the paper is organized as follows. Section II introduces the Gaussian cognitive channel and its high-SNR approximation. In Section III we re-derive the different known outer bounds for the Gaussian cognitive channel in a unified framework inspired by the high-SNR approximation model. In Section IV we develop simple achievable strategies and we show that they are within 1.87 bits from the outer bound for any channel parameters. Section V concludes the paper.

II. Channel Model

We consider the two-user Gaussian Cognitive InterFerence Channel (G-CIFC) in standard form [14] depicted in Fig. 1 whose outputs are

$$Y_1 = X_1 + aX_2 + Z_1$$
$$Y_2 = bX_1 + X_2 + Z_2,$$

where $Z_i \sim \mathcal{N}(0,1)$ and the input X_i is subject to the power constraint $\mathrm{E}[|X_i|^2] \le P_i$, $i = 1,2$. Encoder i wishes to communicate a message W_i uniformly distributed on $\{1,\ldots,2^{NR_i}\}$ to decoder i in N channel uses. The two messages are independent. Encoder 1 (i.e., the cognitive user) in addition to its message W_1, also knows W_2. A rate pair (R_1,R_2) is achievable if there exists a sequence of encoding functions

$$X_1^N = f_1^N(W_1,W_2),\ X_2^N = f_2^N(W_2),\ N = 1,2,...,$$

and a sequence of decoding functions

$$\hat{W}_i = g_i^N(Y_i^N), \quad i = 1,2, \quad N = 1,2,...,$$

such that

$$\max_{i=1,2} \Pr\left[\hat{W}_i \ne W_i\right] \to 0, \quad N \to \infty.$$

In [13] we introduced the deterministic high-SNR approximation of the G-CIFC and determined its capacity. We restate the main result here for completeness. Let

$$n_{ij} = \left\lfloor \frac{1}{2}\log\left(1+|h_{ij}|^2 P_j\right)\right\rfloor, \quad m \triangleq \max_{i,j}\{n_{ij}\},$$

for $h_{11} = h_{22} = 1,\ h_{12} = a,\ h_{21} = b$. The high-SNR approximation of a G-CIFC is a deterministic channel whose outputs are

$$Y_i = \sum_{j=1}^{2} \mathbf{S}^{m-n_{ij}} X_j, \quad i = 1,2,$$

where X_j is a binary vector of dimension m, $\mathbf{S}$ is a shift matrix of dimension $m \times m$ and the summation is taken in GF(2) [6]. The capacity of the deterministic cognitive channel is

$$R_1 \le n_{11} \tag{1a}$$
$$R_2 \le \max\{n_{21}, n_{22}\} \tag{1b}$$
$$R_1 + R_2 \le \max\{n_{21}, n_{22}\} + [n_{11} - n_{21}]^+, \tag{1c}$$

where $[x]^+ \triangleq \max\{0,x\}$. The sum-rate bound in (1c) is obtained by giving receiver 1, the cognitive receiver, the signal $S_1 = \mathbf{S}^{n_{21}} X_1$ as side information. In the next section we re-derive the different known outer bounds for the Gaussian cognitive channel as summarized in [14] in a unified framework inspired by the derivation of (1) in [13].

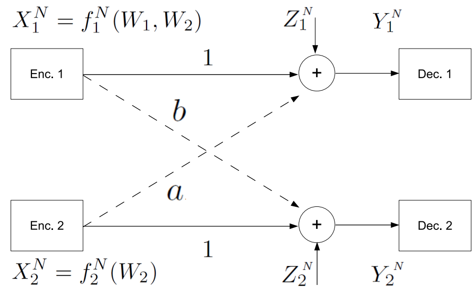

Fig. 1. A G-CIFC in canonical form.

III. Outer Bound

The following summarizes the results of [3], [15], [16]:

Theorem III.1. *The capacity region of a G-CIFC is within the convex-hull of*

$$R_1 \le \frac{1}{2}\log\left(1+(1-\rho^2)P_1\right) \tag{2a}$$
$$R_2 \le \frac{1}{2}\log\left(1+|b|^2P_1+P_2+2\rho\sqrt{|b|^2P_1P_2}\right) \tag{2b}$$
$$R_1+R_2 \le \frac{1}{2}\log\left(1+|b|^2P_1+P_2+2\rho\sqrt{|b|^2P_1P_2}\right) + \frac{1}{2}\log\left(\frac{1+(1-\rho^2)\max\{1,|b|^2\}P_1}{1+(1-\rho^2)|b|^2P_1}\right) \tag{2c}$$

for all $\rho \in [0,1]$.

Proof: For any joint distribution on (X_1,X_2) let (X_{1G},X_{2G}) be a jointly Gaussian input with the same covariance matrix as (X_1,X_2). If R_1 is achievable, then $H(W_1|Y_1^N,W_2) \le H(W_1|Y_1^N) \le N\epsilon_N$, with $\epsilon_N \to 0$ as $N \to \infty$. Similarly, if R_2 is achievable, then $H(W_2|Y_2^N) \le N\epsilon_N$.

Let $\rho\sqrt{P_1P_2} \triangleq \mathrm{E}[X_1X_2]$. The rate of the cognitive user (user 1) is bounded by (2a) since:

$$\begin{aligned}
NR_1 &\le H(W_1) = H(W_1|W_2)\\
&\le I(W_1;Y_1^N|W_2,X_2^N(W_2)) + N\epsilon_N\\
&= -h(Y_1^N|W_2,X_2^N(W_2),W_1,X_1^N(W_1,W_2)) +\\
&\quad + h(Y_1^N|W_2,X_2^N(W_2)) + N\epsilon_N\\
&\le h(Y_1^N|X_2^N) - h(Y_1^N|X_2^N,X_1^N) + N\epsilon_N\\
&= I(Y_1^N;X_1^N|X_2^N) + N\epsilon_N\\
&\le N\Big(I(Y_1;X_{1G}|X_{2G}) + \epsilon_N\Big).
\end{aligned}$$

The rate of the primary user (user 2) is bounded by (2b) since:

$$\begin{aligned} NR_2 &\le H(W_2) \\ &\le I(W_2;Y_2^N) + N\epsilon_N \\ &= h(Y_2^N) - h(Y_2^N|W_2, X_2^N(W_2)) + N\epsilon_N \\ &\le h(Y_2^N) - h(Y_2^N|X_2^N, X_1^N) + N\epsilon_N \\ &= I(Y_2^N; X_1^N, X_2^N) + N\epsilon_N \\ &\le N\Big(I(Y_2; X_{1G}, X_{2G}) + \epsilon_N\Big). \end{aligned}$$

For the sum-rate we use steps similar to those of [17], [13]. Let $S_i \triangleq bX_{1i} + Z'_{2i}$, where (Z_{1i}, Z'_{2i}) is a jointly Gaussian random vector whose entries have zero mean and unit power, and $\mathrm{E}[Z_{1i}Z'_{2i}] = \rho_Z$. The correlation coefficient $\rho_Z \in [-1,1]$ can be chosen to tighten the upper bound. We have:

$$\begin{aligned} N(R_1+R_2) &\le H(W_1|W_2) + H(W_2) \\ &\le I(W_1;Y_1^N|W_2) + I(W_2;Y_2^N) + N2\epsilon_N \\ &\le I(W_1;Y_1^N, S_1^N|W_2) + I(W_2;Y_2^N) + N2\epsilon_N \\ &= h(Y_1^N|W_2, S_1^N, X_2^N(W_2)) + h(S_1^N|W_2) \\ &\quad - h(Y_1^N, S_1^N|W_1, W_2, X_2^N(W_2), X_1^N(W_1,W_2)) \\ &\quad + h(Y_2^N) - h(Y_2^N|W_2) + N2\epsilon_N \\ &\le h(Y_1^N|S_1^N, X_2^N) + h(Y_2^N) - h(Z_1^N, Z_2'^N) + N2\epsilon_N \\ &= I(Y_1^N; X_1^N|S_1^N, X_2^N) + I(Y_2^N; X_1^N, X_2^N) + N2\epsilon_N \\ &\le N\Big(I(Y_1; X_{1G}|bX_{1G} + Z'_{2i}, X_{2G}) \\ &\quad + I(Y_2; X_{1G}, X_{2G}) + 2\epsilon_N\Big), \end{aligned}$$

where we have used the fact that

$$h(S_1^N|W_2, X_2^N(W_2)) = h(Y_2^N|W_2, X_2^N(W_2)).$$

Since the above bound is valid for any ρ_z we conclude that

$$\begin{aligned} R_1 + R_2 &\le \frac{1}{2}\log(1 + P_2 + |b|^2P_1 + 2|\rho|\sqrt{|b|^2P_1\,P_2}) \\ &+ \min_{\rho_z\in[0,1]} \frac{1}{2}\log\left(\frac{1-\rho_Z^2 + (1-\rho^2)P_1(|b|^2+1-2|b|\rho_Z)}{[1-\rho_Z^2][1+|b|^2(1-\rho^2)P_1]}\right) \end{aligned}$$

After substituting the optimal value of ρ_z given by

$$\operatorname*{argmin}_{\rho_z} \frac{|b|^2 + 1 - 2|b|\rho_Z}{1-\rho_Z^2} = \min\left\{|b|, \frac{1}{|b|}\right\}$$

we obtain that the sum-rate is bounded by (2c). ■

Remark III.2. In strong interference ($|b| > 1$) the region in (2) reduces to [14, Th.5] because the bound (2b) is redundant due to (2c). The outer bound in (2) is known to be achievable in *very strong interference*, that is, if $|b| > 1$ and $\alpha\, a \ge \left(\sqrt{\alpha^2 + |b|^2 - 1 + 2\rho\alpha b + \rho^2} - \rho\right)$ where $\alpha = \sqrt{P_1/P_2}$, holds for every $|\rho| \le 1$. In this case the capacity is achieved using a scheme where both receivers decode both messages. Notice that, in strong interference receiver 2 can decode both messages without imposing any rate penalty on the rate of receiver 1. Indeed, after decoding W_2, receiver 2 can reconstruct $X_2^N(W_2)$ and compute the following estimate of the receiver 1 output

$$\frac{Y_2^N - X_2^N}{b} + aX_2^N + \sqrt{1 - \frac{1}{|b|^2}}Z^N \sim Y_1^N$$

where $Z^N \sim \mathcal{N}(0, \mathbf{I})$ and independent of everything else. A similar statement with the role of the users reversed and for $|a| \ge 1$ is not possible since the knowledge of W_1 does not allow the reconstruction of $X_1^N(W_1, W_2)$.

Remark III.3. In weak interference ($|b| \le 1$) the region in (2) reduces to [3, Th. 3.2], [15, Th. 4.1] as the closure of the region is determined by the rates pairs for which (2a) and (2c) are met with equality [18, Ex. 4.3]. In this case the capacity is achieved using a scheme where the primary receiver treats the cognitive message as interference.

In the next section we develop simple achievable strategies that are are within 1.87 bits from the outer bound in (2) for any channel parameters (a, b, P_1, P_2).

IV. Achievable Schemes

A. Weak interference ($|b| < 1$)

The outer bound (2) was shown to be achievable in [3], [15] in weak interference. In this case the achievable scheme employs dirty paper coding (DPC) at the cognitive transmitter to "cancel" the known interference generated by the primary user, and collaboration in the transmission of the primary message in a MISO fashion. At the primary decoder the message of the cognitive user is treated as noise. The interference due the primary user at the receiver of the cognitive user is completely canceled by the use of DPC.

While capacity is known in this regime, here we consider a very simple broadcast strategy that is an alternative strategy to the capacity achieving one of [3], [15] and show that it is optimal to within 1 bit for the case of weak interference ($|b|^2P_1 < P_1$) and weak signal

($|b|^2 P_1 \geq P_2$). In weak interference, the region in (2) can be re-written as [3], [15]

$$R_1 \leq \frac{1}{2}\log\left(1+(1-\rho^2)P_1\right) \tag{3a}$$

$$R_2 \leq \frac{1}{2}\log\left(\frac{1+|b|^2P_1+P_2+2\rho\sqrt{|b|^2P_1P_2}}{1+|b|^2(1-\rho^2)P_1}\right) \tag{3b}$$

for $\rho \in [0,1]$. Consider now a strategy in which transmitter 1 sends information to both receivers and transmitter 2 is silent. This strategy is inspired by the achievability of (1) in [13]. With $X_2=0$ and $|b|<1$ we have a degraded broadcast channel $X_1 \to Y_1 \to Y_2$, whose capacity region is

$$R_1 \leq \frac{1}{2}\log(1+(1-\rho^2)P_1) \tag{4a}$$

$$R_2 \leq \frac{1}{2}\log\left(\frac{1+|b|^2P_1}{1+(1-\rho^2)|b|^2P_1}\right) \tag{4b}$$

for $\rho \in [0,1]$. Then, since (3a) and (4a) are the same for every ρ there is zero gap for the cognitive user (user 1). By considering the difference between (3b) and (4b), the gap for primary user (user 2) is bounded as

$$\begin{aligned}(3\text{b})-(4\text{b}) &\leq \frac{1}{2}\log\left(1+\frac{P_2+2\sqrt{|b|^2P_1P_2}}{1+|b|^2P_1}\right)\\ &\leq \frac{1}{2}\log\left(1+\frac{3|b|^2P_1}{1+|b|^2P_1}\right) \leq \frac{1}{2}\log(4)=1.\end{aligned}$$

This shows that for weak interference ($|b|^2P_1 < P_1$) and weak signal ($|b|^2P_1 \geq P_2$) a superposition coding looses at most 1 bit with respect capacity achieving DPC.

B. *Strong interference* ($|b| \geq 1$)

The corner points of the outer bound region in (2), i.e., the points for which the rate of one user is the maximum possible, are obtained for $\rho=1$ and $\rho=0$. For $\rho=1$, the rate pair

$$A=\left(0, \frac{1}{2}\log\left(1+(\sqrt{|b|^2P_1}+\sqrt{P_2})^2\right)\right)$$

is achievable if transmitter 1 uses all its power to transmit W_2 in cooperation with transmitter 2 in a MISO fashion. For $\rho=0$ we have

$$B=\left(\frac{1}{2}\log\left(1+P_1\right), \frac{1}{2}\log\left(\frac{1+|b|^2P_1+P_2}{1+P_1}\right)\right).$$

Remark IV.1. In strong interference the capacity outer bound in (2) can be further outer bounded by

$$R_1 \leq \frac{1}{2}\log\left(1+P_1\right) \tag{5a}$$

$$R_1+R_2 \leq \frac{1}{2}\log\left(1+(\sqrt{|b|^2P_1}+\sqrt{P_2})^2\right) \tag{5b}$$

In the following we will use the fact that the point B is at most $1/2$ bit away from (5) since

$$\frac{1}{2}\log\left(\frac{1+(\sqrt{|b|^2P_1}+\sqrt{P_2})^2}{1+|b|^2P_1+P_2}\right) \leq \frac{1}{2}\log(2)=\frac{1}{2}.$$

In the following we show achievable schemes that are within a finite gap from the rate pair B. We divide the analysis into two cases.

a) Strong interference ($|b|^2P_1 \geq P_1$) and weak signal ($|b|^2P_1 \geq P_2$): We consider again a simple broadcast strategy from transmitter 1 to both receivers (with transmitter 2 being silent) and we show that it is optimal to within $1/2$ bit per user. This strategy is again inspired by the achievability of (1). With $X_2=0$ and $|b| \geq 1$ we have a degraded broadcast channel $X_1 \to Y_2 \to Y_1$, whose capacity is

$$R_1^{(C)} \leq \frac{1}{2}\log\left(\frac{1+P_1}{1+\alpha P_1}\right) \tag{6a}$$

$$R_2^{(C)} \leq \frac{1}{2}\log\left(1+\alpha|b|^2P_1\right) \tag{6b}$$

for $\alpha \in [0,1]$. If we set $\alpha=\min\{1,1/P_1\}$ in (6), then the gap for user 1 is

$$R_1^{(B)}-R_1^{(C)}=\frac{1}{2}(1+\min\{1,P_1\}) \leq \frac{1}{2}\log 2=\frac{1}{2},$$

while for user 2 (using $P_2 \leq |b|^2P_1$ and $|b|^2 \geq 1$) is

$$\begin{aligned}&R_2^{(B)}-R_2^{(C)} \leq \frac{1}{2}\log\left(\frac{1+2|b|^2P_1}{(1+P_1)(1+|b|^2\min\{1,P_1\})}\right)\\ &\leq \max\left\{\frac{1}{2}\log\left(\frac{2|b|^2}{1+|b|^2}\right), \frac{1}{2}\log\left(\frac{2}{1+P_1}\right)\right\}\\ &\leq \frac{1}{2}\log 2=\frac{1}{2}.\end{aligned}$$

As shown in Fig. 2, the achievable point C in (6) is at most at $1/2+1=1.5$ bits from the outer bound. By time sharing between points A and C, we have an achievable region that is at most at $\max\{0.5,1.5\}=1.5$ bits from the outer bound in (2).

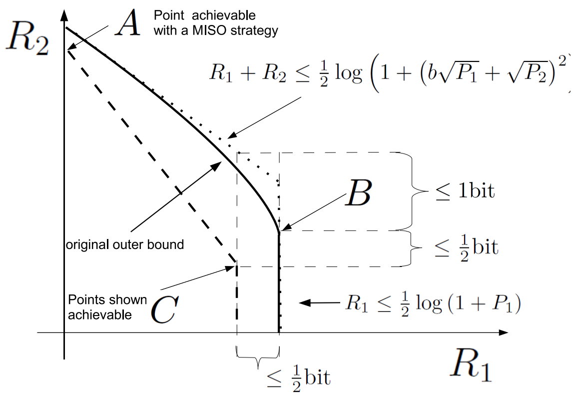

Fig. 2. G-CIFC with strong interference: solid line: outer bound in (2); dashed line: achievable region by time sharing among points A and C; dotted line: outer bound in (5).

b) Strong interference ($|b|^2 P_1 \geq P_1$) and strong signal ($|b|^2 P_1 \leq P_2$): The details of the proof for this case can be found in [19] and are not reported here for sake of space. The proving strategy is as for the previous case. We show an achievable scheme for point C that is within 1.37 (combining the maximum gap for R_1 and the maximum gap for R_2) from point B. Then by time sharing between A and C we have an achievable region that is at most 1.37 +$1/2$ =1.87 bits from outer bound in (5). The achievable scheme is from [5] and employs DPC at the cognitive receiver and joint decoding at the primary receiver only. Although this achievable scheme does not reduce to superposition coding, which is capacity achieving in the very strong interference as shown in [16], it is within a finite gap from the capacity for any channel parameter.

V. Conclusions

Inspired by the recent results on the high-SNR deterministic approximation of the Gaussian cognitive radio channel, we presented simple achievable schemes producing an achievable rate region that lies within 1.87 of a new generalized capacity outer bound.

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