A class of Bi-directional multi-relay protocols

Sang Joon Kim, Natasha Devroye, and Vahid Tarokh

***Abstract*— In a bi-directional relay channel, two nodes wish to exchange independent messages over a shared wireless half-duplex channel with the help of relays. Recent work has considered information theoretic limits of the bi-directional relay channel with a single relay. In this work we consider bi-directional relaying with multiple relays. We derive achievable rate regions and outer bounds for half-duplex protocols with multiple decode and forward relays and compare these to the same protocols with amplify and forward relays in an additive white Gaussian noise channel. We consider three novel classes of half-duplex protocols: the $(m, 2)$ 2 phase protocol with m relays, the $(m, 3)$ 3 phase protocol with m relays, and general (m, t) Multiple Hops and Multiple Relays (MHMR) protocols, where m is the total number of relays and $3 < t \leq m + 2$ is the number of temporal phases in the protocol. Finally, we provide a comprehensive treatment of the MHMR protocols with decode and forward relaying and amplify and forward relaying in Gaussian noise, obtaining their respective achievable rate regions, outer bounds and relative performance at different SNRs. The $(m, m + 2)$ DF MHMR protocol achieves the largest rate region under simulated channel conditions.**

***Index Terms*— bi-directional communication, achievable rate regions, decode and forward, amplify and forward, multiple relays**

I. Introduction

In bi-directional channels, two *terminal nodes* (a and b) wish to exchange independent messages. In wireless channels or mesh networks, this communication may take place with the help of m other nodes r_i, $i \in \{1, 2, \cdots m\}$ termed relays. This two-way channel [2] was first considered in [9], in which full-duplex operation where nodes could transmit and receive simultaneously, was assumed. Since full-duplex operation is, with current technology, of limited practical significance, in this work we assume that the nodes are *half-duplex*, i.e. at each point in time, a node can either transmit or receive symbols, but not both.

Our main goal is to determine the limits of bi-directional communication with multiple relays. To do so, we propose and determine the achievable rate regions, as well as outer bounds obtained using several protocols. The protocols we propose for the multiple-relay bi-directional channel may be described in terms of two parameters: the number of relays, m, and the number of temporal phases t, called *hops*. Throughout this work, *phases* and *hops* are used interchangeably. We also define an *intermediate hop* as a hop in which only relays transmit (and not the terminal nodes). Note that our protocols are all composed of a number of temporal phases/hops due to the half-duplex nature of the channel. We denote our proposed protocols as (m, t) MHMR (Multiple Hops and Multiple Relays) protocols, for general positive integers $m \geq 2$ and $t \geq 2$. For the special case of two hops ($t = 2$), the terminal nodes may simultaneously transmit in phase 1 as in the MABC (Multiple Access Broadcast Channel) protocol of [5], while the relays transmit the decoded messages to the terminal nodes in phase 2. For the special case of three hops ($t = 3$) the terminal nodes may sequentially transmit in the first two phases as in the TDBC (Time Division Broadcast Channel) protocol of [5], after which the relays transmit in phase 3. In the TDBC protocol, side information may be exploited to improve the rate region. By side information we mean information obtained from the wireless channel in a particular phase which may be combined with information obtained in different stages to potentially improve decoding or increase transmission rates.

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For each of the MHMR protocols, the relays may process and forward the received signals differently. Standard forwarding techniques include decode-and-forward, amplify-and-forward, compress-and-forward, and de-noise and forward. We consider only the first two relaying schemes.

Some similar protocols and relaying schemes have been previously considered. In [6], the DF TDBC protocol with a single relay is considered. There, network coding in $\mathbb{Z}_2^k$ is used to encode the message of relay r from the estimated messages $\tilde{w}_{\mathsf{a}}$ and $\tilde{w}_{\mathsf{b}}$. The works [7] and [8] consider the MABC protocol with multiple hops, where an amplification and denoising relaying scheme is introduced. In [5], achievable rate regions and outer bounds of the MABC protocol and the TDBC protocol for a single DF relay are derived. In [1], a comprehensive analysis of the AF scheme in large networks is provided.

This paper is structured as follows: in Section II, we introduce our notation. In Section III, we introduce novel (m, t) MHMR protocols. In Section IV we derive achievable rate regions for the (m, t) MHMR protocols with DF relaying. In Section V we derive outer bounds for the MHMR protocols. In Section VI, we numerically compute these bounds in the Gaussian noise channel and compare the results for different powers and channel conditions.

II. Preliminaries

Nodes a and b are the two terminal nodes and $\mathcal{R} := \{\mathsf{r}_1, \mathsf{r}_2, \cdots, \mathsf{r}_m\}$ is the set of relays which aid the communication between nodes a and b. For convenience of analysis we define $\mathsf{r}_0 := \mathsf{a}$, $\mathsf{r}_{m+1} := \mathsf{b}$ and use these notations interchangeably. Also define $\mathcal{R}^* := \mathcal{R} \cup \{\mathsf{a}, \mathsf{b}\} = \{\mathsf{r}_0, \mathsf{r}_1, \cdots, \mathsf{r}_{m+1}\}$. We use $R_{i,j}$ for the transmitted data rate from node i to node j. In our case, two terminal nodes denoted a and b exchange their messages W_{a} and W_{b} at the rates $R_{\mathsf{a}} := R_{\mathsf{a},\mathsf{b}}$ and $R_{\mathsf{b}} := R_{\mathsf{b},\mathsf{a}}$, that is, the two messages W_{a} and W_{b} are taken to be independent and uniformly distributed in the set of $\{0, \cdots, \lfloor 2^{nR_{\mathsf{a}}} \rfloor - 1\} := \mathcal{S}_{\mathsf{a}}$ and $\{0, \cdots, \lfloor 2^{nR_{\mathsf{b}}} \rfloor - 1\} := \mathcal{S}_{\mathsf{b}}$.

Each node i has channel input alphabet $\mathcal{X}_i^* = \mathcal{X}_i \cup \{\varnothing\}$ and channel output alphabet $\mathcal{Y}_i^* = \mathcal{Y}_i \cup \{\varnothing\}$. Because of the half-duplex constraint, not all nodes transmit/receive during all phases and we use the dummy symbol $\varnothing$ to denote that there is no input or no output at a particular node during a particular phase. The half-duplex constraint forces either $X_i^{(\ell)} = \varnothing$ or $Y_i^{(\ell)} = \varnothing$ for all ℓ phases. The channel is assumed to be discrete *memoryless*. For convenience, we drop the notation $\varnothing$ from entropy and mutual information terms when a node is not transmitting or receiving. During phase ℓ we use $X_i^{(\ell)}$ to denote the input distribution and $Y_i^{(\ell)}$ to denote the distribution of the received signal of node i and similarly $X_S^{(\ell)} := \{X_i^{(\ell)} | i \in S\}$, a set of input distributions during phase ℓ. Δ_i is the phase duration of phase i when $n \to \infty$. Lower case letters x_i denote instances of the upper case X_i which lie in the calligraphic alphabets $\mathcal{X}_i^*$. $\prod_{i=1}^N \mathcal{X}_i$ denotes the Cartesian product of the alphabets $\mathcal{X}_i$, $i = 1, 2, \cdots, N$ for non-negative integer N. Boldface $\mathbf{x}_i$ represents a vector indexed by time at node i.

For the $(m, 2)$, $(m, 3)$ DF MHMR protocols we define $\mathcal{A}$ (resp. $\mathcal{B}$) as the set of relays which are able to decode w_{a} (resp. w_{b}). We define $I_S^{\min}(X_i^{(\ell)}; Y_s^{(\ell)}) := \min_{s \in S} I(X_i^{(\ell)}; Y_s^{(\ell)})$. For example, $I_{\mathcal{A}}^{\min}(X_{\mathsf{a}}^{(1)}; Y_{\mathsf{r}}^{(1)}) = \min_{\mathsf{r} \in \mathcal{A}} I(X_{\mathsf{a}}^{(1)}; Y_{\mathsf{r}}^{(1)})$, i.e. the minimum mutual information between node a and a relay in the set of relays which can decode w_{a}. We also define $I_{\varnothing}^{\min}(X;Y) = 0$. **$\bigotimes$ denotes the cartesian product, i.e., $\bigotimes_{i=1}^3 \mathcal{X}_i = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$.**

III. Protocols

We next describe a class of bi-directional multiple-relay protocols which we term (m, t) *DF MHMR* (Decode and Forward, Multiple Hop Multiple Relay) protocols, where m is the number of relays and t is the number of hops. A protocol is a series of temporal phases through which bi-directional communication between nodes a and b is enabled. In Section VI we consider Amplify and Forward relaying in the Gaussian channel and use the term (m, t) *AF MHMR* protocol. We rename MHMR protocols for some special cases as follow:

- MABC MHMR protocol : $(m, 2)$ MHMR protocol
- TDBC MHMR protocol : $(m, 3)$ MHMR protocol [1]
- *regular* MHMR protocol : each hop has the same number of relays, only valid when $m \mod (t-2) = 0$. [2]

We first describe the $(m, m+2)$ MHMR protocol. From the $(m, m+2)$ MHMR protocol the (m, t) for $3 < t < m+2$ protocol and corresponding achievable rate regions readily follow.

$$\mathsf{r}_0 (= \mathsf{a}) \leftrightarrows \mathsf{r}_1 \leftrightarrows \mathsf{r}_2 \leftrightarrows \cdots \leftrightarrows \mathsf{r}_m \leftrightarrows \mathsf{r}_{m+1} (= \mathsf{b})$$

is one possible graphical representation of our multi-hop network with m relays. A simple naïve protocol for the above example network is : $\mathsf{r}_0 \to \mathsf{r}_1 \to \cdots \to \mathsf{r}_m \to \mathsf{r}_{m+1}$ and then $\mathsf{r}_{m+1} \to \mathsf{r}_m \to \cdots \to \mathsf{r}_1 \to \mathsf{r}_0$. This is one possible $(m, 2m+2)$ MHMR protocol which may be spectrally inefficient as the number of phases is large. Intuitively, spectral efficiency may be improved by combining phases through the use of network coding. We reduce the number of phases needed from $2m+2$ to $m+2$ by the algorithm in Table I.

TABLE I

[ALGORITHM] - $(m, m+2)$ DF MHMR protocol

Preparation

a and b divide their respective messages into B sub-messages, one for each *block*. Thus, a has the message set $\{w_{\mathsf{a},(0)}, w_{\mathsf{a},(1)}, \cdots, w_{\mathsf{a},(B-1)}\}$. Likewise b has $\{w_{\mathsf{b},(0)}, w_{\mathsf{b},(1)}, \cdots, w_{\mathsf{b},(B-1)}\}$.

Initialization

For $i = 0$ to $m-1$
  For $j = 0$ to i
    r_{i-j} transmits $\mathbf{x}_{\mathsf{r}_{i-j}}(w_{\mathsf{a},(j)})$
    r_{i-j+1} decodes $w_{\mathsf{a},(j)}$
  end
end

Main routine

For $i = 0$ to $B-m-1$
  r_{m+1} transmits $\mathbf{x}_{\mathsf{r}_{m+1}}(w_{\mathsf{b},(i)})$
  r_m decodes $w_{\mathsf{b},(i)}$ and generates $\mathbf{x}_{\mathsf{r}_m}(w_{\mathsf{a},(i)} \oplus w_{\mathsf{b},(i)})$
  For $j = 0$ to $m-1$
    r_{m-j} transmits $\mathbf{x}_{\mathsf{r}_{m-j}}(w_{\mathsf{a},(i+j)} \oplus w_{\mathsf{b},(i)})$
    r_{m-j-1} decodes $w_{\mathsf{b},(i)}$ and generates $\mathbf{x}_{\mathsf{r}_{m-j-1}}(w_{\mathsf{a},(i+j+1)} \oplus w_{\mathsf{b},(i)})$
    r_{m-j+1} decodes $w_{\mathsf{a},(i+j)}$
  end
  r_0 transmits $\mathbf{x}_{\mathsf{r}_0}(w_{\mathsf{a},(m+i)})$
  r_1 decodes $w_{\mathsf{a},(m+i)}$
end

Termination

For $i = B-m$ to $B-1$
  r_{m+1} transmits $\mathbf{x}_{\mathsf{r}_{m+1}}(w_{\mathsf{b},(i)})$
  r_m decodes $w_{\mathsf{b},(i)}$ and generates $\mathbf{x}_{\mathsf{r}_m}(w_{\mathsf{a},(i)} \oplus w_{\mathsf{b},(i)})$
  For $j = 0$ to $m-1$
    r_{m-j} transmits $\mathbf{x}_{\mathsf{r}_{m-j}}$
    r_{m-j-1} decodes $w_{\mathsf{b},(i)}$ and generates
$$\begin{cases} \mathbf{x}_{\mathsf{r}_{m-j-1}}(w_{\mathsf{a},(i+j+1)} \oplus w_{\mathsf{b},(i)}), & \text{if } i+j \le B-2 \\ \mathbf{x}_{\mathsf{r}_{m-j-1}}(w_{\mathsf{b},(i)}), & \text{otherwise} \end{cases}$$
    r_{m-j+1} decodes $w_{\mathsf{a},(i+j)}$ if $i+j \le B-1$
  end
end

After initialization, relay r_i has the following messages from node a: $\{w_{\mathsf{a},(0)}, w_{\mathsf{a},(1)}, \cdots, w_{\mathsf{a},(m-i)}\}$ $(1 \le i \le m)$. In other words message $w_{\mathsf{a},(i)}$ has reached r_{m-i} at the end of the initialization. In the main routine, which, when the number of blocks $B \to \infty$ makes up the majority of this protocol, $w_{\mathsf{b},(i)}$ travels along the path $\mathsf{r}_{m+1} \to \mathsf{r}_m \to \cdots \to \mathsf{r}_1 \to \mathsf{r}_0$ in the i^{th} loop. During the same loop, as the single sub-message from node b travels to node a, the stream of messages from node a sitting in the each of the relays are all shifted to the right by one through the use of network coding. Overall then, we require 2 transmissions from the terminal nodes, and m relay transmissions to transfer two individual sub-messages. When node a finishes sending its all sub-messages to r_1, the termination step starts. The remaining $w_{\mathsf{a},(i)}$s in the relays and $w_{\mathsf{b},(i)}$s in node b are processed in this step. The number of transmissions in the main routine depends only on the number of blocks B while the number of transmissions in

[1] The relaying protocols for the MABC and TDBC protocols are described in [4] and we do not state them here due to space constraints. The protocols are a simple extension of the single relay protocols in [5].

[2] For example, the $(8, 6)$ regular MHMR protocol consists of two relays in each of the four hops.

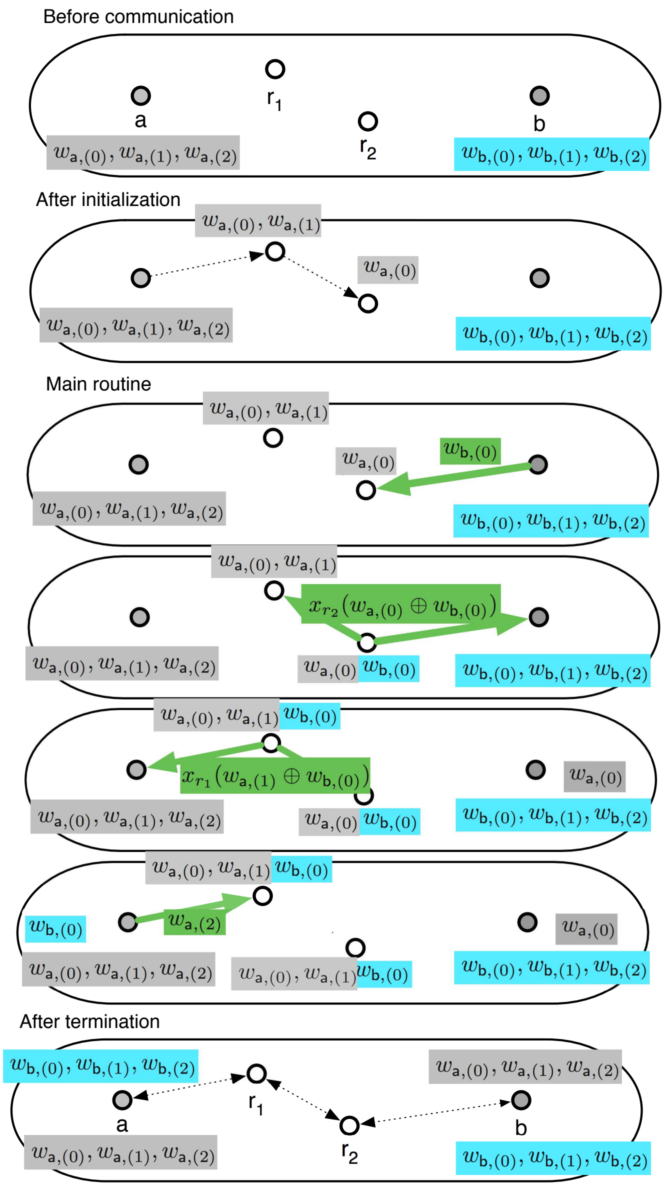

Fig. 1. Illustration of the $(m, m+2)$ DF MHMR protocol with B=3, m=2. Grey denotes the sub-messages of w_a at the nodes, blue denotes the sub-messages of w_b at the nodes, and green denotes the current transmission. Dotted lines denote the path taken during initialization and termination phases.

the initialization and termination steps are a function of the hop size m. We can easily show that by increasing the block size B, our algorithm asymptotically results in $m+2$ phases. A graphical illustration for the case when $B = 3$ and $m = 2$ is shown in Fig. 1.

IV. Achievable Rate Regions

Remark : Due to space constraints, the proofs for the Theorems 1, 2, 3, and Corollary 4 are provided in [4].

A. $(m, 2)$ DF MABC protocol

Theorem 1: An achievable region of the half-duplex bi-directional channel under the $(m, 2)$ DF MABC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\Big\{\Delta_1 I^{\min}_{\mathcal{A}\cap\mathcal{B}}(X_a^{(1)}; Y_r^{(1)}|X_b^{(1)}, Q), \Delta_1 I^{\min}_{\mathcal{A}\setminus\mathcal{B}}(X_a^{(1)}; Y_r^{(1)}|Q), \Delta_2 I(X_{\mathcal{A}}^{(2)}; Y_b^{(2)}|Q)\Big\} \quad (1)$$

$$R_b < \min\Big\{\Delta_1 I^{\min}_{\mathcal{A}\cap\mathcal{B}}(X_b^{(1)}; Y_r^{(1)}|X_a^{(1)}, Q), \Delta_1 I^{\min}_{\mathcal{B}\setminus\mathcal{A}}(X_b^{(1)}; Y_r^{(1)}|Q), \Delta_2 I(X_{\mathcal{B}}^{(2)}; Y_a^{(2)}|Q)\Big\} \quad (2)$$

$$R_a + R_b < \Delta_1 I^{\min}_{\mathcal{A}\cap\mathcal{B}}(X_a^{(1)}, X_b^{(1)}; Y_r^{(1)}|Q) \quad (3)$$

over all joint distributions $p(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)$ $p^{(2)}(x_{\mathcal{A}\cap\mathcal{B}}|q)p^{(2)}(x_{\mathcal{A}\setminus\mathcal{B}}|q)p^{(2)}(x_{\mathcal{B}\setminus\mathcal{A}}|q)$ with $|\mathcal{Q}| \leq 3m+2$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{r_i}$ for all possible $\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}$. □

B. $(m, 3)$ DF TDBC protocol

Theorem 2: An achievable region of the half-duplex bi-directional channel under the $(m, 3)$ DF TDBC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\Big\{\Delta_1 I^{\min}_{\mathcal{A}}(X_a^{(1)}; Y_r^{(1)}|Q), \Delta_1 I(X_a^{(1)}; Y_b^{(1)}|Q) + \Delta_3 I(X_{\mathcal{A}}^{(3)}; Y_b^{(3)}|Q)\Big\} \quad (4)$$

$$R_b < \min\Big\{\Delta_2 I^{\min}_{\mathcal{B}}(X_b^{(2)}; Y_r^{(2)}|Q), \Delta_2 I(X_b^{(2)}; Y_a^{(2)}|Q) + \Delta_3 I(X_{\mathcal{B}}^{(3)}; Y_a^{(3)}|Q)\Big\} \quad (5)$$

over all joint distributions $p(q)p^{(1)}(x_a|q)p^{(2)}(x_b|q)$ $p^{(3)}(x_{\mathcal{A}\cap\mathcal{B}}|q)p^{(3)}(x_{\mathcal{A}\setminus\mathcal{B}}|q)p^{(3)}(x_{\mathcal{B}\setminus\mathcal{A}}|q)$ with $|\mathcal{Q}| \leq 2m+2$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{r_i}$ for all possible $\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}$. □

C. (m, t) DF MHMR protocol

First, we recall that in the $(m, m+2)$ MHMR protocol a single relay transmits in each hop. We then extend the ideas of the $(m, m+2)$ MHMR protocol to derive achievable rate regions for general (m, t) protocols with $3 < t < m+2$.

Theorem 3: An achievable rate region of the half-duplex bi-directional multi-hop relay channel under the $(m, m+2)$ DF MHMR protocol $(m > 1)$ is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min_{1\leq k\leq m+1} \left\{ \sum_{i=1}^{k} \Delta_{m+3-i} I(X_{r_{i-1}}^{(m+3-i)}; Y_{r_k}^{(m+3-i)}|Q) \right\} \quad (6)$$

$$R_b < \min_{1\leq k\leq m+1} \left\{ \sum_{i=1}^{k} \Delta_i I(X_{r_{m+2-i}}^{(i)}; Y_{r_{m+1-k}}^{(i)}|Q) \right\} \quad (7)$$

over all joint distributions $p(q)\prod_{i=1}^{m+2} p^{(i)}(x_{r_{m+2-i}}|q)$ with $|\mathcal{Q}| \leq 2m+2$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{r_i}$. □

Corollary 4: An achievable rate region of the half-duplex bi-directional channel in the (m, t) DF MHMR protocol for $3 < t < m+2$ is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min_{1\leq k\leq t-1} \min_{r_k\in\mathcal{R}_k} \left\{ \sum_{i=1}^{k} \Delta_{t+1-i} I(X_{\mathcal{R}_{i-1}}^{(t+1-i)}; Y_{r_k}^{(t+1-i)}|Q) \right\} \quad (8)$$

$$R_b < \min_{1\leq k\leq t-1} \min_{r_{t-1-k}\in\mathcal{R}_{t-1-k}} \left\{ \sum_{i=1}^{k} \Delta_i I(X_{\mathcal{R}_{t-i}}^{(i)}; Y_{r_{t-1-k}}^{(i)}|Q) \right\} \quad (9)$$

over all joint distributions $p(q)\prod_{i=1}^{t} p^{(i)}(x_{\mathcal{R}_{t-i}}|q)$ with $|\mathcal{Q}| \leq 2m+2$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{r_i}$, for all possible $\mathcal{R}_i \subset \mathcal{R}$ such that $\mathcal{R}_i \cap \mathcal{R}_j = \varnothing$ for all $i, j \in [0, t-1]$, where $\mathcal{R}_0 = \{a\}$ and $\mathcal{R}_{t-1} = \{b\}$. □

V. Outer Bounds

Remark : **We derive outer bounds using the cut-set bound lemma in [5].** Due to space constraints, the proofs for the Theorems 5, 6, 7, and Corollary 8 are provided in [4].

A. $(m,2)$ MABC protocol

Theorem 5: (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the $(m,2)$ MABC protocol is outer bounded by the set of rate pairs $(R_{\mathsf{a}}, R_{\mathsf{b}})$ satisfying

$$R_{\mathsf{a}} \leq \min_{S_R} \left\{ \Delta_1 I(X_{\mathsf{a}}^{(1)}; Y_{\bar{S}_R}^{(1)} | X_{\mathsf{b}}^{(1)}, Q) + \Delta_2 I(X_{S_R}^{(2)}; Y_{\mathsf{b}}^{(2)} | X_{\bar{S}_R}^{(2)}, Q) \right\} \tag{10}$$

$$R_{\mathsf{b}} \leq \min_{S_R} \left\{ \Delta_1 I(X_{\mathsf{b}}^{(1)}; Y_{\bar{S}_R}^{(1)} | X_{\mathsf{a}}^{(1)}, Q) + \Delta_2 I(X_{S_R}^{(2)}; Y_{\mathsf{a}}^{(2)} | X_{\bar{S}_R}^{(2)}, Q) \right\} \tag{11}$$

for all choices of the joint distribution $p(q)p^{(1)}(x_{\mathsf{a}}|q)$ $p^{(1)}(x_{\mathsf{b}}|q)p^{(2)}(x_{S_R}|q)$ with $|\mathcal{Q}| \leq 2^{m+1}$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{\mathsf{r}_i}$ for all possible $S_R \subseteq \mathcal{R}$. □

B. $(m,3)$ TDBC protocol

Theorem 6: (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the $(m,3)$ TDBC protocol is outer bounded by the set of rate pairs $(R_{\mathsf{a}}, R_{\mathsf{b}})$ satisfying

$$R_{\mathsf{a}} \leq \min_{S_R} \left\{ \Delta_1 I(X_{\mathsf{a}}^{(1)}; Y_{\bar{S}_R}^{(1)}, Y_{\mathsf{b}}^{(1)} | Q) + \Delta_3 I(X_{S_R}^{(3)}; Y_{\mathsf{b}}^{(3)} | X_{\bar{S}_R}^{(3)}, Q) \right\} \tag{12}$$

$$R_{\mathsf{b}} \leq \min_{S_R} \left\{ \Delta_2 I(X_{\mathsf{b}}^{(2)}; Y_{\bar{S}_R}^{(2)}, Y_{\mathsf{a}}^{(2)} | Q) + \Delta_3 I(X_{S_R}^{(3)}; Y_{\mathsf{a}}^{(3)} | X_{\bar{S}_R}^{(3)}, Q) \right\} \tag{13}$$

for all choices of the joint distribution $p(q)p^{(1)}(x_{\mathsf{a}}|q)$ $p^{(2)}(x_{\mathsf{b}}|q)p^{(3)}(x_{S_R}|q)$ with $|\mathcal{Q}| \leq 2^{m+1}$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{\mathsf{r}_i}$ for all possible $S_R \subseteq \mathcal{R}$. □

C. (m,t) MHMR protocol

Theorem 7: (Outer bound) The capacity region of the half-duplex bi-directional multi-hop relay channel under the $(m, m+2)$ MHMR protocol $(m > 1)$ is outer bounded by the set of rate pairs $(R_{\mathsf{a}}, R_{\mathsf{b}})$ satisfying

$$R_{\mathsf{a}} \leq \min_{S_R} \left\{ \sum_{\mathsf{r}_i \in S_R \cup \{\mathsf{a}\}} \Delta_{m+2-i} I(X_{\mathsf{r}_i}^{(m+2-i)}; Y_{\bar{S}_R}^{(m+2-i)}, Y_{\mathsf{b}}^{(m+2-i)} | Q) \right\} \tag{14}$$

$$R_{\mathsf{b}} \leq \min_{S_R} \left\{ \sum_{\mathsf{r}_i \in S_R \cup \{\mathsf{b}\}} \Delta_{m+2-i} I(X_{\mathsf{r}_i}^{(m+2-i)}; Y_{\bar{S}_R}^{(m+2-i)}, Y_{\mathsf{a}}^{(m+2-i)} | Q) \right\} \tag{15}$$

for all choices of the joint distribution $p(q)$ $\prod_{i=1}^{m+2} p^{(i)}(x_{\mathsf{r}_{m+2-i}}|q)$ with $|\mathcal{Q}| \leq 2^{m+1}$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{\mathsf{r}_i}$ for all possible $S_R \subseteq \mathcal{R}$. □

Corollary 8: (Outer bound) The capacity region of the half-duplex bi-directional channel in the (m,t) MHMR protocol for $3 < t < m+2$ is outer bounded by the set of rate pairs $(R_{\mathsf{a}}, R_{\mathsf{b}})$ satisfying

$$R_{\mathsf{a}} \leq \min_{S_R} \left\{ \sum_{i=0}^{t-1} \Delta_{t-i} I(X_{\mathcal{R}_i \cap (S_R \cup \{\mathsf{a}\})}^{(t-i)}; Y_{\bar{S}_R \setminus \mathcal{R}_i}^{(t-i)}, Y_{\mathsf{b}}^{(t-i)} | X_{\bar{S}_R \cap \mathcal{R}_i}^{(t-i)}, Q) \right\} \tag{16}$$

$$R_{\mathsf{b}} \leq \min_{S_R} \left\{ \sum_{i=0}^{t-1} \Delta_{t-i} I(X_{\mathcal{R}_i \cap (S_R \cup \{\mathsf{b}\})}^{(t-i)}; Y_{\bar{S}_R \setminus \mathcal{R}_i}^{(t-i)}, Y_{\mathsf{a}}^{(t-i)} | X_{\bar{S}_R \cap \mathcal{R}_i}^{(t-i)}, Q) \right\} \tag{17}$$

for all choices of the joint distribution $p(q) \prod_{i=1}^{t} p^{(i)}(x_{\mathcal{R}_{t-i}}|q)$ with $|\mathcal{Q}| \leq 2^{m+1}$ over the restricted alphabet $\bigotimes_{i=0}^{m+1} \mathcal{X}_{\mathsf{r}_i}$, for all possible $\mathcal{R}_i \subset \mathcal{R}$ such that $\mathcal{R}_i \cap \mathcal{R}_j = \varnothing$ for all $i, j \in [0, t-1]$, where $\mathcal{R}_0 = \{\mathsf{a}\}$ and $\mathcal{R}_{t-1} = \{\mathsf{b}\}$ for all possible $S_R \subseteq \mathcal{R}$. □

VI. Numerical Analysis

A. The Gaussian relay network

In this section, we apply the bounds obtained in the previous section to a Gaussian relay network. The corresponding mathematical channel model is, for each channel use k :

$$\mathbf{Y}[k] = \mathbf{H}\mathbf{X}[k] + \mathbf{Z}[k] \tag{18}$$

where $\mathbf{Y}[k]$, $\mathbf{X}[k]$ and $\mathbf{Z}[k]$ are in $(\mathbb{C} \cup \{\varnothing\})^{(m+2)\times 1}$, and $\mathbf{H} \in \mathbb{C}^{(m+2)\times(m+2)}$. In phase ℓ, if node r_i is in transmission mode $X_{\mathsf{r}_i}[k]$ follows the input distribution $X_{\mathsf{r}_i}^{(\ell)} \sim \mathcal{N}(0, P_{\mathsf{r}_i})$. Otherwise, $X_{\mathsf{r}_i}[k] = \varnothing$, which means that the input symbol does not exist in the above mathematical channel model.

In each phase, the total transmit power is bounded by P. While ideally the per-phase power of P could be distributed amongst the nodes in $\mathcal{R}_\ell$ arbitrarily, as a first step, we allocate equal power $P/|\mathcal{R}_\ell|$ for each relay in $\mathcal{R}_\ell$. Equal power allocation between participating nodes may also be simpler to implement. Then, in each phase, we allow for cooperation between relays which have the same messages. $h_{i,j}$ is the effective channel gain between transmitter i and receiver j. We assume the channel is reciprocal $(h_{i,j} = h_{j,i})$ and that each node is fully aware of the channel gains, i.e., full CSI. The noise at all receivers is independent, of unit power, additive, white Gaussian, complex and circularly symmetric.

As a comparison point for the DF MHMR protocols, we derive an achievable region of the same temporal protocols in which the relays use a simple amplify and forward relaying scheme rather than a decode and forward scheme. “Simple” means that there is no power optimization in each phase, i.e. each node during phase ℓ has equal transmit power $P/|\mathcal{R}_\ell|$. Also, in the amplify and forward scheme, all phase durations are equal since relaying is performed on a symbol by symbol basis. Thus, $\Delta_\ell = \frac{1}{t}$, where t is the number of phases and $\ell \in [1, t]$. Furthermore, relay r scales the received symbol by $\sqrt{\frac{P_r}{P_{y_{\mathsf{r}}}}}$ to meet the transmit power constraint. [3]

B. Rate region comparisons with one to two relays

In this section we numerically evaluate the rate regions in the Gaussian relay network with two relays. We use the following channel gain matrix [4]:

$$\mathbf{H} = \begin{bmatrix} 0 & 1.2 & 0.8 & 0.2 \\ 1.2 & 0 & 2 & 0.8 \\ 0.8 & 2 & 0 & 1.2 \\ 0.2 & 0.8 & 1.2 & 0 \end{bmatrix} \tag{19}$$

In the proposed protocols, the (2,4) DF MHMR protocol achieves the largest rate region in most scenarios. In the high

[3] Due to space constraints, achievable rate regions for AF MHMR protocols are provided in [4].

[4] If other channel gains are chosen, the numerical results may change.

SNR regime, the (2,2) AF MABC protocol may achieve rates slightly better than the (2,4) DF MHMR protocol, as noise amplification is less of an issue. **However, in most cases multiple hops with DF relaying dominates in this bi-directional half-duplex channel.** In the DF relaying protocols, the (2,4) MHMR protocol outperforms the other protocols at both low and high SNR. This improved performance may be attributed to this protocol's effective use of side information. During each phase, every node which is not transmitting can receive the current transmission which it may employ as side information to aid decoding during later stages. There is naturally a tradeoff between the number of phases and the amount of information broadcasted in each phase. However, as seen by our simulations in this particular channel, the effect of reducing the number of phases to 2 or 3 does not outweigh the effect of broadcasting information.

The inner and outer bounds differ for a number of reasons, with the prevailing one being that our inner bounds use a DF scheme. For the MABC scheme using DF relaying, every relay contributes to enlarging the outer bound regions, while only the subset of relays $\mathcal{A}\cup\mathcal{B}$ are used in determining the achievable regions. At low SNR, when $\mathcal{A}\cup\mathcal{B}$ is relatively small, the gaps, shown in Fig. 2 (top) are larger than the gaps at high SNRs shown in Fig. 2 (bottom), where the number of relays in $\mathcal{A}\cup\mathcal{B}$ are relatively larger. In addition to simply having more relays contribute to the outer bound regions, their effect is summed up outside of the logarithm for the outer bound, and inside of it for the inner bounds. Lastly, the achievable rate regions for DF relaying are significantly reduced by the necessity of having all relays decode the message(s) w_{a} or w_{b} individually, resulting in the min function which significantly diminishes the region. This requirement to decode all messages is not present in the outer bounds. The inner bounds for the AF relaying schemes are relatively small as (a) noise is carried forward, (b) no power optimization is performed and (c) no phase-length optimization is performed. The inner bounds may be improved through the use of compress and forward relaying [3] or de-noising, which may be able to capture the optimal tradeoff between eliminating the noise and requiring the messages to be decoded. The exploration of different relaying schemes as well as the analytical impact of different channel gain matrices is left for future work.

Due to space constraints we defer the interested reader to [4] for a rate region comparison in which 8 relays are placed uniformly on a line. There, the $(8,10)$ DH MHMR protocol dominates the other protocols both in the low SNR and high SNR regime. This may be attributed to the broadcasted side information: while increasing the number of phases means that less information may be transmitted during each time phase, the accumulated side information and improved channel gains (shorter distances) for each hop outweigh these detrimental effects, yielding higher overall rates.

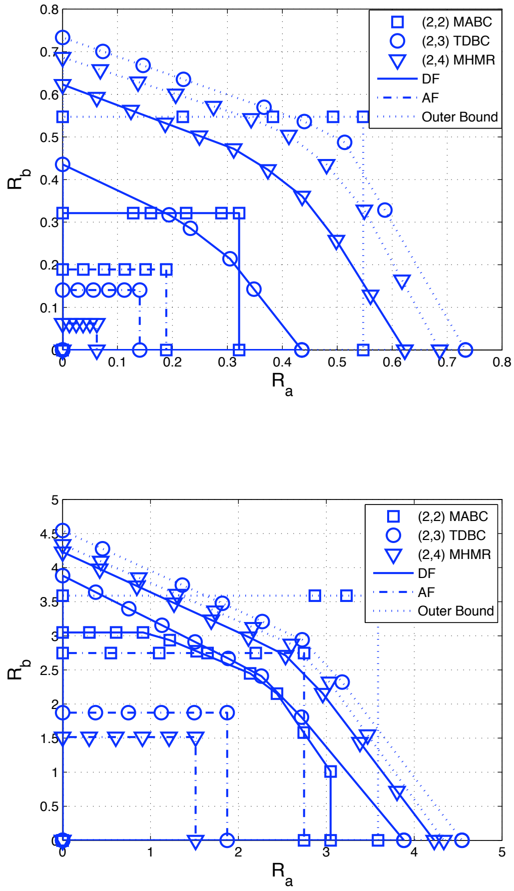

Fig. 2. Comparison of achievable regions of AF and DF and outer bounds with $P = 0$ dB (top) and $P = 20$ dB (bottom).

VII. Conclusion

In this paper, we proposed protocols for the half-duplex bi-directional channel with multiple relays. We derived achievable rate regions as well as outer bounds for 3 half-duplex bi-directional multiple relay protocols with decode and forward relays. We compared these regions to those achieved by the same protocols with amplify and forward relays in the Gaussian noise channel. Numerical evaluations suggest that the $(m, m+2)$ DF MHMR protocol achieves the largest rate region under simulated channel conditions.

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