List decoding for nested lattices and applications to relay channels

Yiwei Song, Natasha Devroye
Lattices codes good for multi-user AWGN channels

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- AWGN inference channel: interference decoding / interference alignment in K>2 interference channels [Bresler, Parekh, Tse, ArXiv 2008] [Sridharan, Jafarian, Jafar, Shamai, arXiv 2008]
Lattices codes for two-way AWGN relay channels

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  [Wilson, Narayanan, Pfister, Sprintson, Trans. IT, to appear] [Nam, Chung, Lee, Trans. IT, to appear]
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  \[ R \]

  - achieve to within 1/2 bit/s/Hz gap of cut-set outer bound in absence of direct link
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- AWGN multi-way relay channels

  \[ X \]

  - achieve to within 2 bit/sez/Hz/user gap of cut-set outer bound in absence of direct links
Lattice codes missing in?

- AWGN relay channel?
Lattice codes missing in?

- AWGN relay channel?

- Two-way relay channel in **presence of direct links**?
Contributions

• **List decoder** for nested lattices:
  *decode a list of particular size which contains correct codeword*
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- **One-way relay channel:** *use list decoder to achieve capacity of physically degraded AWGN relay channel with nested lattices*
Contributions

- **List decoder** for nested lattices: decode a list of particular size which contains correct codeword

- **One-way relay channel**: use list decoder to achieve capacity of physically degraded AWGN relay channel with nested lattices

- **Two-way relay channel**: use list decoder to obtain new achievable rate region and finite gap results for “degraded” cases of two-way relay channel with direct links
Lattice notation

- \( \Lambda = \{ \lambda = G \mathbf{i} : \mathbf{i} \in \mathbb{Z}^n \} \), \( G \) the generator matrix
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- lattice quantizer of $\Lambda$:
  \[ Q(X) = \arg \min_{\lambda \in \Lambda} ||X - \lambda|| \]
Lattice notation

- $\Lambda = \{\lambda = G\, i : i \in \mathbb{Z}^n\}$, $G$ the generator matrix

- **lattice quantizer of $\Lambda$:**
  $$Q(X) = \arg\min_{\lambda \in \Lambda} ||X - \lambda||$$

- $x \mod \Lambda := x - Q(x)$
Lattice notation

- \( \Lambda = \{ \lambda = G \mathbf{i} : \mathbf{i} \in \mathbb{Z}^n \} \), \( G \) the generator matrix

- *lattice quantizer* of \( \Lambda \):
  \[
  Q(\mathbf{X}) = \arg \min_{\lambda \in \Lambda} ||\mathbf{X} - \lambda||
  \]

- \( \mathbf{x} \mod \Lambda := \mathbf{x} - Q(\mathbf{x}) \)

- *fundamental region* \( \mathcal{V} := \{ \mathbf{x} : Q(\mathbf{x}) = \mathbf{0} \} \) of volume \( V \)
Lattice notation

- $\Lambda = \{\lambda = G \mathbf{i} : \mathbf{i} \in \mathbb{Z}^n\}$, $G$ the generator matrix

- **lattice quantizer of $\Lambda$:**
  
  $$Q(\mathbf{X}) = \arg\min_{\lambda \in \Lambda} ||\mathbf{X} - \lambda||$$

- $\mathbf{x} \mod \Lambda := \mathbf{x} - Q(\mathbf{x})$

- **fundamental region** $\mathcal{V} := \{\mathbf{x} : Q(\mathbf{x}) = \mathbf{0}\}$ of volume $V$

- **second moment per dimension of a uniform distribution over $\mathcal{V}$:**
  
  $$\sigma^2(\Lambda) := \frac{1}{V} \cdot \frac{1}{n} \int_{\mathcal{V}} ||\mathbf{x}||^2 d\mathbf{x}$$
Nested lattice codes

• Nested lattice pair: $\Lambda \subseteq \Lambda_c$ ( $\Lambda$ is Rogers-good and Poltyrev-good, $\Lambda_c$ is Poltyrev-good)
Nested lattice codes

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- The code book \( \mathcal{C} = \{ \Lambda_c \cap \mathcal{V}(\Lambda) \} \) is used to achieve the capacity of AWGN channel

\[ \Lambda \subseteq \Lambda_c \]

\[ \mathcal{C} = \{ \Lambda_c \cap \mathcal{V}(\Lambda) \} \]

\[ [\text{Erez+Zamir, Trans. IT, 2004}] \]
Nested lattice codes

- Nested lattice pair: $\Lambda \subseteq \Lambda_c$ ( $\Lambda$ is Rogers-good and Poltyrev-good, $\Lambda_c$ is Poltyrev-good )

- The code book $\mathcal{C} = \{\Lambda_c \cap \mathcal{V}(\Lambda)\}$ is used to achieve the capacity of AWGN channel [Erez+Zamir, Trans. IT, 2004]

- Coding rate: $R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log \frac{V(\Lambda)}{V(\Lambda_c)}$ arbitrary
Nested lattice chains

- $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_K$ ( $\Lambda_1, \Lambda_2 \ldots \Lambda_{K-1}$ are Rogers-good and Poltyrev-good, $\Lambda_K$ is Poltyrev-good ). The nesting rates between any pairs in the chain can attain any arbitrary values as the dimension $n \to \infty$.

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- A "good" lattice chain with length 3 is used in our list decoding scheme:
  
  $\Lambda \subseteq \Lambda_s \subseteq \Lambda_c$

[Image of nested lattice chains]
Lattice list decoder

- IDEA: decode to rather than to
Lattice list decoder

• IDEA: decode to \( \Lambda \subseteq \Lambda_s \subseteq \Lambda_c \) rather than to

• results in a list of codewords
Lattice list decoder

- IDEA: decode to a list rather than to

- results in a list of codewords

- require correct codeword to be in list
Lattice list decoder

• IDEA: decode to \( \Lambda \) rather than to \( \Lambda_s \)

• results in a list of codewords

• require correct codeword to be in list

• how many are in list?
Encoding

• message of rate $R$ over the AWGN channel $Y = X + Z$ subject to the average power constraint $P$
Encoding

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- **Encoding:** take $t \in C_{\Lambda, \nu}$ associated with message of rate $R$ and $X = (t - U) \text{ mod } \Lambda$

- $U$ is a dither signal uniformly distributed over $\nu$. 
Decoding

\[ \Lambda \subseteq \Lambda_s \subseteq \Lambda_c \]
Decoding

\[ \Lambda \subseteq \Lambda_s \subseteq \Lambda_c \]

\[ S_{\Lambda_s, \Lambda_c}(Y') = \{ \Lambda_c \cap (Y' + \mathcal{N}_s) \} \]

\[ \star = Y' \]
Decoding

• Receiver first computes

\[ Y' = (\alpha Y + U) \mod \Lambda \]
\[ = (t - (1 - \alpha)X + \alpha Z) \mod \Lambda \]
\[ = (t + Z') \mod \Lambda \]
Decoding

• Receiver first computes

\[ Y' = (\alpha Y + U) \mod \Lambda \]
\[ = (t - (1 - \alpha)X + \alpha Z) \mod \Lambda \]
\[ = (t + Z') \mod \Lambda \]

• Receiver then decodes the list of codewords \( \hat{t} \):

\[ L(\hat{t}) := S_{\mathcal{V}_s,\Lambda_c}(Y') \mod \Lambda \]

\[ \mathcal{S}_{\mathcal{V}_s,\Lambda_c}(Y') = \{ \Lambda_c \cap (Y' + \mathcal{V}_s) \} \]

\[ \star = Y' \]
Lattice list decoder

- Probability of error for list decoding: \( P_e := \Pr\{t \notin L(\hat{t})\} \)
Lattice list decoder

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\[
\begin{align*}
\Lambda &\subseteq \Lambda_s \subseteq \Lambda_c \\
S_{\Lambda_s, \Lambda_c}(Y') &= \{\Lambda_c \cap (Y' + \mathcal{V}_s)\} \\
\star &= Y' \\
S_{\Lambda_s, \Lambda_c}(Y') &= \{\Lambda_c \cap (Y' + \mathcal{V}_s)\} \\
Q_{\mathcal{V}_s, \Lambda_c}(Y') &= \bigcup_{\lambda_c \in \Lambda_c} \{\lambda_c | Y' \in (\lambda_c + \mathcal{V}_s)\}
\end{align*}
\]

- easy to count \# in list

Thursday, September 30, 2010
Lattice list decoder

- Probability of error for list decoding: $P_e := \Pr\{t \notin L(\hat{t})\}$

- easy to count # in list
- easy to bound probability of error
Lattice list decoder

• Theorem 1: Using the encoding and decoding scheme defined above, the receiver decodes a list of codewords of size $2^{n(R-C(P/N))}$ with probability of error $P_e \to 0$ as $n \to \infty$
Application I: AWGN degraded relay channel
(Decode and Forward)

\[ Y_R = X_1 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R) \]

Power \( P_R \)
No message, just relay

\[ Y_2 = X_1 + X_R + Z_2, \quad Z_2 \sim \mathcal{N}(0, N_2) \]

Degraded if \( Z_2 = Z_R + Z'_2 \)
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- Capacity of degraded AWGN relay channel shown to be

\[
R \leq \max_{0 \leq \alpha \leq 1} \min \left\{ \frac{1}{2} \log \left( 1 + \frac{P + P_R + 2\sqrt{\alpha PP_R}}{N + N_R} \right), \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right) \right\}
\]
Application I: AWGN degraded relay channel (Decode and Forward)

- in [Cover, El Gamal, Trans. IT, 1979] proven using:
  - superposition coding
  - Slepian-Wolf partitioning
  - coding for cooperative multiple access channel
  - Block-Markov coding
  - list decoding
  - successive decoding
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all using RANDOM codes
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 **all using RANDOM codes**

 **can we use NESTED LATTICE codes instead?**
Source node (Node 1) sends the superposition of $X_1$ and $X_2$.

\[ X_1 \leftrightarrow \Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1} \]

\[ X_2 \leftrightarrow \Lambda_2 \subseteq \Lambda_{c2} \]

(block Markov coding)
Source node (Node 1) sends the superposition of $X_1$ and $X_2$.

**Encoding**

$X_1 \iff \Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1}$

$X_2 \iff \Lambda_2 \subseteq \Lambda_{c2}$

$E(X_1^2) = \alpha P$

$E(X_2^2) = \bar{\alpha} P$

*block Markov coding*
Source node (Node 1) sends the superposition of $X_1$ and $X_2$

$$E(X_1^2) = \alpha P$$
$$E(X_2^2) = \bar{\alpha} P$$

$$\Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1}$$
$$\sigma^2(\Lambda_1) = \alpha P$$
$$\sigma^2(\Lambda_2) = \bar{\alpha} P$$

(block Markov coding)
Source node (Node 1) sends the superposition of $X_1$ and $X_2$

$E(X_1^2) = \alpha P$  
$\sigma^2(\Lambda_1) = \alpha P$  
List decoding lattice

$E(X_2^2) = \bar{\alpha} P$  
$\sigma^2(\Lambda_2) = \bar{\alpha} P$

Relay node (Node R) sends $X_R$

$X_R = \sqrt{\frac{P_R}{\bar{\alpha} P}} X_2$  
$\leftrightarrow \sqrt{\frac{P_R}{\bar{\alpha} P}} \Lambda_2 \subseteq \sqrt{\frac{P_R}{\bar{\alpha} P}} \Lambda_{c2}$

(block Markov coding)
Encoding

Source node (Node 1) sends the superposition of $X_1$ and $X_2$

$E(X_1^2) = \alpha P$

$\sigma^2(\Lambda_1) = \alpha P$  List decoding lattice

$E(X_2^2) = \bar{\alpha} P$

$\sigma^2(\Lambda_2) = \bar{\alpha} P$

Relay node (Node R) sends $X_R$

$X_R = \sqrt{\frac{P_R}{\bar{\alpha} P}} X_2$

$\sigma^2(\sqrt{\frac{P_R}{\bar{\alpha} P}} \Lambda_2) = P_R$

(block Markov coding)
Decoding

- At relay: $Y_R = X_1 + X_2 + Z_R$
Decoding

- At relay: $Y_R = X_1 + X_2 + Z_R \quad R < \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_R}\right)$
Decoding

- At relay: \( Y_R = X_1 + X_2 + Z_R \quad R < \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right) \)

- At destination:

\[
Y_2 = X_1 + X_2 + X_R + Z_2 \\
= \left( 1 + \sqrt{\frac{P_R}{\bar{\alpha} P}} \right) X_2 + X_1 + Z_2
\]
Decoding

• At relay: \( Y_R = X_1 + X_2 + Z_R \) \[ R < \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right) \]

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Y_2 = X_1 + X_2 + X_R + Z_2
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\[
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Successive decoding:
Decoding

- At relay: $Y_R = X_1 + X_2 + Z_R \quad R < \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_R}\right)$

- At destination:

$$Y_2 = X_1 + X_2 + X_R + Z_2 = \left(1 + \sqrt{\frac{P_R}{\alpha P}}\right)X_2 + X_1 + Z_2$$

Successive decoding:

Same Slepian-wolf partitioning index (coherent gain)
Decoding

- At relay: $Y_R = X_1 + X_2 + Z_R \quad R < \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right)$

- At destination:

\[
Y_2 = X_1 + X_2 + X_R + Z_2 = \left( 1 + \sqrt{\frac{PR}{\alpha P}} \right) X_2 + X_1 + Z_2
\]

Successive decoding:

Same Slepian-wolf partitioning index (coherent gain)

List decoded
Decoding

- At relay: $Y_R = X_1 + X_2 + Z_R \quad R < \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_R}\right)$

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Successive decoding:

Same Slepian-wolf partitioning index (coherent gain) \quad \rightarrow \quad \text{List decoded}
Decoding

- At relay: \( Y_R = X_1 + X_2 + Z_R \) \[ R < \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right) \]

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Y_2 = X_1 + X_2 + X_R + Z_2 \\
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Successive decoding:

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List decoded
Decoding

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= \left(1 + \sqrt{\frac{P_R}{\bar{\alpha} P}}\right) X_2 + X_1 + Z_2
\]

Successive decoding:

Same Slepian-wolf partitioning index (coherent gain)

List decoded

Desired message
Decoding

- At relay: $Y_R = X_1 + X_2 + Z_R \quad R < \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_R}\right)$

- At destination:

\[ Y_2 = X_1 + X_2 + X_R + Z_2 \]
\[= \left(1 + \sqrt{\frac{P_R}{\alpha P}}\right) X_2 + X_1 + Z_2 \]

Successive decoding:

Same Slepian-wolf partitioning index (coherent gain)

List decoded

$R < \frac{1}{2} \log \left(1 + \frac{P + P_R + 2\sqrt{\alpha P P_R}}{N + N_R}\right)$
Application I: AWGN degraded relay channel

- Capacity of degraded AWGN relay channel shown to be

\[ R \leq \max_{0 \leq \alpha \leq 1} \min \left\{ \frac{1}{2} \log \left( 1 + \frac{P + P_R + 2\sqrt{\alpha P P_R}}{N + N_R} \right), \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right) \right\} \]
Application I: AWGN degraded relay channel

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\[ R \leq \max_{0 \leq \alpha \leq 1} \min \left\{ \frac{1}{2} \log \left( 1 + \frac{P + P_R + 2\sqrt{\alpha PP_R}}{N + N_R} \right), \frac{1}{2} \log \left( 1 + \frac{\alpha P}{N_R} \right) \right\} \]

**Thm 2:** This can be achieved using **NESTED LATTICE CODES**!
Application 2: two-way relay channel with direct links
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- achieve to within 1/2 bit/s/Hz gap in absence of direct link
  [Nam, Chung, Lee, Trans. IT, to appear]
Application 2: two-way relay channel with direct links

- achieve to within $1/2 \text{ bit/s/Hz}$ gap in absence of direct link
  [Nam, Chung, Lee, Trans. IT, to appear]

- uses lattice codes to "decode the sum" at the relay, rather than individual messages
Application 2: two-way relay channel with direct links

\[ Y_r = X_1 + X_2 + N_R \]

- achieve to within 1/2 bit/s/Hz gap in **absence of direct link**
  
  [Nam, Chung, Lee, Trans. IT, to appear]

- achieve to within 2 bits/s/Hz gap for special cases **with direct link**
  
  [Avestimehr, Sezgin, Tse, European Trans. Comm, 2009]

- uses lattice codes to ``decode the sum'' at the relay, rather than individual messages

- uses random codes and quantizers at relay
Application 2: two-way relay channel with direct links
Application 2: two-way relay channel with direct links

\[ Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R) \]

Power \( P_R \)
No message, just relay

\[ Z_1 \sim \mathcal{N}(0, N_1) \]
\[ Z_2 \sim \mathcal{N}(0, N_2) \]

Degraded if \( Z_1 = Z_R + Z_1', Z_2 = Z_R + Z_2' \)
Application 2: two-way relay channel with direct links

\[ Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R) \]

- Power \( P_R \)
- No message, just relay

\[ Y_1 = X_1 + X_2 + X_R + Z_1 \quad Y_2 = X_1 + X_2 + X_R + Z_2 \]

\[ Z_1 \sim \mathcal{N}(0, N_1) \quad Z_2 \sim \mathcal{N}(0, N_2) \]

Degraded if \( Z_1 = Z_R + Z'_1, \quad Z_2 = Z_R + Z'_2 \)

- we derive a new achievable rate region using nested lattices, with direct link
Application 2: two-way relay channel with direct links

\[ Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R) \]

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Degraded if \( Z_1 = Z_R + Z_1', \quad Z_2 = Z_R + Z_2' \)

- we derive a new achievable rate region using nested lattices, with direct link
- this region attains constant gaps for certain degraded channel
Application 2: two-way relay channel with direct links

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Power \( P_R \)
No message, just relay

\[ Y_1 = X_1 + X_2 + X_R + Z_1 \quad Y_2 = X_1 + X_2 + X_R + Z_2 \]

\[ Z_1 \sim \mathcal{N}(0, N_1) \quad Z_2 \sim \mathcal{N}(0, N_2) \]

Degraded if \( Z_1 = Z_R + Z'_1, \quad Z_2 = Z_R + Z'_2 \)

• we derive a new achievable rate region using nested lattices, with direct link

• this region attains constant gaps for certain degraded channel
Application 2: two-way relay channel with direct links

- Random binning
  [Xie, CWIT 2007]
  [Kramer, Shamai, ITW 2007]
Application 2: two-way relay channel with direct links

- Random binning
  [Xie, CWIT 2007]
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- Decoding sum
  [Nam, Chung, Lee Trans. IT to appear]
Application 2: two-way relay channel with direct links

- Random binning
  [Xie, CWIT 2007]
  [Kramer, Shamai, ITW 2007]

\[ T = (t_1 + t_2 - Q_2(t_2 + U_2)) \mod \Lambda_1 \]

- Decoding sum
  [Nam, Chung, Lee
  Trans. IT to appear]

- List decoding
  [this work]

\[ S_{\mathcal{V}, \Lambda_c} (Y') = \{ \Lambda_c \cap (Y' + \mathcal{V}) \} \]

\( \star = Y' \)
Application 2: two-way relay channel with direct links

- Random binning
  - [Xie, CWIT 2007]
  - [Kramer, Shamai, ITW 2007]

\[ w_1, w_2 \]
\[ \hat{w}_2, w_1, \hat{w}_1, w_2 \]

- Decoding sum
  - [Nam, Chung, Lee, Trans. IT to appear]

\[ \hat{T} = (t_1 + t_2 - Q_2(t_2 + U_2)) \mod \Lambda_1 \]

\[ w_1 \leftrightarrow t_1 \]
\[ w_2 \leftrightarrow t_2 \]

- List decoding
  - [this work]

\[ \Lambda \subseteq \Lambda_c \subseteq \Lambda_e \]
\[ S_{\gamma', \Lambda_c} (Y') = \{ \Lambda_c \cap (Y' + \mathcal{V}), \} \]
\[ \star = Y' \]
Outline of achievability scheme

- assume WLOG \( P_1 \geq P_2 \)
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- assume WLOG $P_1 \geq P_2$

$(\Lambda_1, \Lambda_{c1})$

$(block \ Markov\ coding)$
Outline of achievability scheme

- assume WLOG $P_1 \geq P_2$

$w_1 \leftrightarrow t_1 \ (\Lambda_1, \Lambda_{c1}) \ 1 \ 2 \ (\Lambda_2, \Lambda_{c2}) \  w_2 \leftrightarrow t_2$

*(block Markov coding)*
Outline of achievability scheme

- assume WLOG \( P_1 \geq P_2 \)

- decode \( \hat{T} = (t_1 + t_2 - Q_2(t_2 + U_2)) \mod \Lambda_1 \)

\[
\begin{align*}
\text{block Markov coding}
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Outline of achievability scheme

- assume WLOG $P_1 \geq P_2$
  - decode $\hat{T} = (t_1 + t_2 - Q_2(t_2 + U_2)) \mod \Lambda_1$
  - send bin index of $\hat{T}$ (random code)

$$w_1 \leftrightarrow t_1 \quad (\Lambda_1, \Lambda_{c1})$$
$$w_2 \leftrightarrow t_2 \quad (\Lambda_2, \Lambda_{c2})$$

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- Relay node: \( Y_R = X_1 + X_2 + Z_R \), decodes \( \hat{T} \):

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R_1 < \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right)
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- Analogous for node 1
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R_1 \leq \min \left( \left[ \frac{1}{2} \log \left( \frac{P_1}{P_1 + P_2} + \frac{P_1}{N_R} \right) \right]^+, \frac{1}{2} \log \left( 1 + \frac{P_1 + P_R}{N_2} \right) \right)
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• eliminates “MAC”-like constraints at relay

• combines direct and relayed information using lattice list decoder
Finite-gap results

\[ Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R) \]

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- Two-way stochastically degraded: \( N_1 \geq N_R \) AND \( N_2 \geq N_R \):
  \[ \frac{1}{2} \log 3 \text{ bit gap.} \]
Numerical evaluations

• Comparison with other Decode-and-Forward schemes which utilize the direct

  [Rankov, Wittneben, ISIT 2006]
  [Xie CWIT 2007]
  Cut-set
Conclusions / questions

- Thm 1: lattice list decoder
- Thm 2: lattices achieve the capacity of the physically degraded relay channel
- Thm 3: new achievable rate region for two-way relay channel *with direct links*
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Questions?