On the Capacity of Multi-user Two-way Linear Deterministic Channels

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Most wireless communications are two-way
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Information Theory: point-to-point two-way channel

[Claude. E. Shannon, 1961]
Information Theory: point-to-point two-way channel

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Capacity in general remains open.
Why two-way problem is so hard?
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Adaptation!
Why two-way problem is so hard?

Adaptation!

\[
x_1^i(m_{12}, y_1^{i-1}) \quad \rightarrow \quad \text{Channel} \quad \leftarrow \quad x_2^i(m_{21}, y_2^{i-1})
\]

\[
y_1^i \quad \rightarrow \quad y_2^i \quad \leftarrow \quad y_2^{i-1} = (y_{2,1}, y_{2,2}, \ldots, y_{2,i-1})
\]
Why two-way problem is so hard?

Adaptation!

Binary Multiplier Channel:  

\[ Y_1 = Y_2 = X_1 X_2 \]

\[ x_1, x_2, y_1, y_2 \in \{0, 1\} \]
Why two-way problem is so hard?

Adaptation!

Binary Multiplier Channel: \( Y_1 = Y_2 = X_1 X_2 \)

\( x_1^i(m_{12}, y_1^{i-1}) \) \hspace{1cm} \( y_1^i \)

\( x_2^i(m_{21}, y_2^{i-1}) \) \hspace{1cm} \( y_2^i \)

\( y_2^{i-1} = (y_{2,1}, y_{2,2}, \ldots, y_{2,i-1}) \)

Adaptation is useful and capacity is unknown.
Two-way deterministic modulo 2 adder channel

\[ X_1(M_{12}) \rightarrow Y_1 \leftarrow M_{21} \]

\[ X_2(M_{21}) \rightarrow Y_2 \rightarrow M_{12} \]

\[ X_1, X_2, Y_1, Y_2 \in \{0, 1\} \]

\[ Y_1 = Y_2 = X_1 \oplus X_2 \]
Two-way deterministic modulo 2 adder channel

\[ X_1 = Y_1 = X_1 \oplus X_2 \]
\[ Y_2 \] = \[ X_2 \]

At each channel use,
\[ Y_1 = X_1 \oplus X_2 \]
\[ Y_2 = X_1 \oplus X_2 \]
Two-way deterministic modulo 2 adder channel

At each channel use,

\[ Y_1 = X_1 \oplus X_2 \oplus X_1 \]
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Two-way deterministic modulo 2 adder channel

\[ Y_1 = Y_2 = X_1 \oplus X_2 \]

\[ x_1, x_2, y_1, y_2 \in \{0, 1\} \]

At each channel use,

\[ Y_1 = X_1 \oplus X_2 \oplus X_1 = X_2 \]
\[ Y_2 = X_1 \oplus X_2 \oplus X_2 = X_1 \]
Two-way deterministic modulo 2 adder channel

\[
Y_1 = Y_2 = X_1 \oplus X_2
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At each channel use,

\[
Y_1 = X_1 \oplus X_2 \oplus X_1 = X_2
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Two-way deterministic modulo 2 adder channel

\[ Y_1 = Y_2 = X_1 \oplus X_2 \]

\[ X_1, X_2, Y_1, Y_2 \in \{0, 1\} \]

\[ Y_1 = X_1 \oplus X_2 \oplus X_1 = X_2 \]

\[ Y_2 = X_1 \oplus X_2 \oplus X_2 = X_1 \]

Capacity region = 1 bit / channel use / user
Two-way deterministic modulo 2 adder channel

\[ Y_1 = Y_2 = X_1 \oplus X_2 \]

\[ x_1, x_2, y_1, y_2 \in \{0, 1\} \]

At each channel use,
\[ Y_1 = \overline{X_2 \oplus X_1} = X_2 \]
\[ Y_2 = X_1 \oplus \overline{X_2} \oplus X_2 = X_1 \]

Capacity region = 1 bit / channel use / user

Adaptation is not needed.
Gaussian two-way point-to-point channel

\[ Y_1 = aX_1 + bX_2 + N_1 \]
\[ Y_2 = cX_1 + dX_2 + N_2 \]
\[ N_1, N_2 \sim \mathcal{N}(0, \sigma^2) \]

\( a, b, c \) and \( d \) are channel gains.
Gaussian two-way point-to-point channel

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\( a, b, c \) and \( d \) are channel gains.

[\text{T. Han, 1984}]
Gaussian two-way point-to-point channel

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\( a, b, c \) and \( d \) are channel gains.

Again, adaptation does not increase capacity.
What about two-way networks?
3 channel models we consider

(a) Two-way MAC/BC
3 channel models we consider

(a) Two-way MAC/BC

(b) Two-way Z channel
3 channel models we consider

(a) Two-way MAC/BC

(b) Two-way Z channel

(c) Two-way interference channel
3 channel models we consider

(a) Two-way MAC/BC

(b) Two-way Z channel

(c) Two-way interference channel

Consider linear deterministic model [Avestimehr; Diggavi; Tse; 2007]
Two-way Linear Deterministic MAC/BC

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23}) \]

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \]

\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]

\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3, \]

\( S \) shift matrix, inputs binary vectors
Two-way Linear Deterministic MAC/BC

\[
N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23})
\]

\[
Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2
\]

\[
Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3
\]

\[
Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3,
\]

$S$ shift matrix, inputs binary vectors
Two-way Linear Deterministic MAC/BC

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Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \]
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Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[
Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3, \]

\(S\) shift matrix, inputs binary vectors

\(N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23})\)
Two-way Linear Deterministic MAC/BC

\[ M_{12} \quad \overline{M_{21}} \]

\[ M_{32} \quad \overline{M_{23}} \]

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \]

\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]

\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 \]

\[ S \text{ shift matrix, inputs binary vectors} \]

Self-interference

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23}) \]
Two-way Linear Deterministic MAC/BC

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \]
\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3, \]

\( N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23}) \)

Theorem 1: The capacity region of the two-way linear deterministic MAC/BC is the set of non-negative rate tuples \((R_{12}, R_{32}, R_{21}, R_{23})\) such that

\[
\text{MAC} \rightarrow \begin{cases} 
R_{12} \leq n_{12}, & R_{32} \leq n_{32}, \\
R_{12} + R_{32} \leq \max(n_{12}, n_{32}) 
\end{cases} \quad (1)
\]

\[
\text{BC} \leftarrow \begin{cases} 
R_{21} \leq n_{21}, & R_{23} \leq n_{23} \\
R_{21} + R_{23} \leq \max(n_{21}, n_{23}). 
\end{cases} \quad (2)
\]
Two-way Linear Deterministic MAC/BC

Adaptation useless (does not increase capacity region)!

![Diagram showing nodes and connections]

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23}) \]
\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \]
\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3, \]

*S shift matrix, inputs binary vectors*

*Theorem 1:* The capacity region of the two-way linear deterministic MAC/BC is the set of non-negative rate tuples \((R_{12}, R_{32}, R_{21}, R_{23})\) such that

\[
\text{MAC} \rightarrow \begin{cases} 
R_{12} \leq n_{12}, & R_{32} \leq n_{32}, \\
R_{12} + R_{32} \leq \max(n_{12}, n_{32}) 
\end{cases}
\] (1)

\[
\text{BC} \leftarrow \begin{cases} 
R_{21} \leq n_{21}, & R_{23} \leq n_{23} \\
R_{21} + R_{23} \leq \max(n_{21}, n_{23}) 
\end{cases}
\] (2)
Two-way Linear Deterministic MAC/BC

**Adaptation useless (does not increase capacity region)!**

\[
N = \max(n_{11}, n_{22}, n_{33}, n_{12}, n_{21}, n_{32}, n_{23})
\]

\[
Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2
\]

\[
Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3
\]

\[
Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3
\]

*S shift matrix, inputs binary vectors

**Achievability:** cancel ``self-interference'' and use non-adaptive one-way schemes

**Converse:** cut-set or alternative
Two-way Linear Deterministic Z Channel

Theorem 2: The capacity region of the two-way linear deterministic Z channel is the set of all rate-tuples \((R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})\) which satisfy the following:

\[
Z \rightarrow \begin{cases} 
R_{12} \leq n_{12}, & R_{32} \leq n_{32}, & R_{34} \leq n_{34} \\
R_{12} + R_{32} \leq \max(n_{12}, n_{32}) \\
R_{32} + R_{34} \leq \max(n_{32}, n_{34}) \\
R_{12} + R_{32} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ 
\end{cases}
\]

\[
Z \leftarrow \begin{cases} 
R_{43} \leq n_{43}, & R_{23} \leq n_{23}, & R_{21} \leq n_{21} \\
R_{43} + R_{23} \leq \max(n_{43}, n_{23}) \\
R_{23} + R_{21} \leq \max(n_{23}, n_{21}) \\
R_{43} + R_{23} + R_{21} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ 
\end{cases}
\]

\[
Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \\
Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \\
Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \\
Y_4 = S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \\
N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{34}, n_{43})
\]
Two-way Linear Deterministic Z Channel

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \]
\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]
\[ Y_4 = S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]

**Theorem 2**: The capacity region of the two-way linear deterministic Z channel is the set of all rate-tuples \((R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})\) which satisfy the following:

\[
Z \rightarrow \begin{cases} 
R_{12} \leq n_{12}, & R_{32} \leq n_{32}, & R_{34} \leq n_{34} \\
R_{12} + R_{32} \leq \max(n_{12}, n_{32}) \\
R_{32} + R_{34} \leq \max(n_{32}, n_{34}) \\
R_{12} + R_{32} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ 
\end{cases}
\]

\[
Z \leftarrow \begin{cases} 
R_{43} \leq n_{43}, & R_{23} \leq n_{23}, & R_{21} \leq n_{21} \\
R_{43} + R_{23} \leq \max(n_{43}, n_{23}) \\
R_{23} + R_{21} \leq \max(n_{23}, n_{21}) \\
R_{43} + R_{23} + R_{21} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+. 
\end{cases}
\]
Two-way Linear Deterministic Z Channel

Theorem 2: The capacity region of the two-way linear deterministic Z channel is the set of all rate-tuples \((R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})\) which satisfy the following:

\[
Z \rightarrow \begin{cases} 
R_{12} \leq n_{12}, & R_{32} \leq n_{32}, & R_{34} \leq n_{34} \\
R_{12} + R_{32} \leq \max(n_{12}, n_{32}) \\
R_{32} + R_{34} \leq \max(n_{32}, n_{34}) \\
R_{12} + R_{32} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ 
\end{cases}
\]

\[
Z \leftarrow \begin{cases} 
R_{43} \leq n_{43}, & R_{23} \leq n_{23}, & R_{21} \leq n_{21} \\
R_{43} + R_{23} \leq \max(n_{43}, n_{23}) \\
R_{23} + R_{21} \leq \max(n_{23}, n_{21}) \\
R_{43} + R_{23} + R_{21} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+. 
\end{cases}
\]

\[
Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \\
Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \\
Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \\
Y_4 = S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 
\]
Two-way Linear Deterministic Z Channel

\[ \begin{align*}
M_{12} &\quad R_{12} \quad M_{12} \quad \hat{M}_{32} \\
\overline{M}_{21} &\quad R_{21} \quad \overline{M}_{21} \quad M_{23} \\
\hat{M}_{32} &\quad R_{32} \quad R_{34} \quad \overline{M}_{34} \\
\overline{M}_{23} &\quad R_{23} \quad R_{43} \quad \overline{M}_{43}
\end{align*} \]

\[ \begin{align*}
Y_1 &= S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 \\
Y_2 &= S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \\
Y_3 &= S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \\
Y_4 &= S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4
\end{align*} \]

**Theorem 2:** The capacity region of the two-way linear deterministic Z channel is the set of all rate-tuples \((R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})\) which satisfy the following:

\[ \begin{align*}
Z &\to \begin{cases}
R_{12} \leq n_{12}, & R_{32} \leq n_{32}, & R_{34} \leq n_{34} \\
R_{12} + R_{32} \leq \max(n_{12}, n_{32}) \\
R_{32} + R_{34} \leq \max(n_{32}, n_{34}) \\
R_{12} + R_{32} + R_{34} \leq \max(n_{12}, n_{32}) + \lfloor n_{34} - n_{32} \rfloor^+ 
\end{cases}
\]

\[ \begin{align*}
Z &\leftarrow \begin{cases}
R_{43} \leq n_{43}, & R_{23} \leq n_{23}, & R_{21} \leq n_{21} \\
R_{43} + R_{23} \leq \max(n_{43}, n_{23}) \\
R_{23} + R_{21} \leq \max(n_{23}, n_{21}) \\
R_{43} + R_{23} + R_{21} \leq \max(n_{43}, n_{23}) + \lfloor n_{21} - n_{23} \rfloor^+. 
\end{cases}
\]
Fano as in deterministic Z channel

\[ n(R_{12} + R_{32} + R_{34} - \epsilon) \leq I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{43}) + I(M_{32}, M_{34}; Y_2^n, Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}) \]
\[ \leq H(Y_2^n | M_{21}, M_{23}, M_{43}) + H(Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}, Y_2^n) \]
\[ \leq \sum_{i=1}^{n} [H(Y_{2,i}|Y_{2,i-1}, M_{21}, M_{23}, X_2^i) + H(Y_{4,i}|M_{12}, M_{21}, M_{23}, M_{43}, Y_{4,i-1}, X_4^i, Y_2^n, X_2^n)] \]
\[ \leq \sum_{i=1}^{n} [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i}) \]
\[ + H(S^{N-n_{34}} X_{3,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_{4,i-1}, X_4^i, S^{N-n_{12}} X_{1,i} + S^{N-n_{22}} X_2,i + S^{N-n_{32}} X_{3,i}, X_2^n, X_1^n)] \]
\[ \leq \sum_{i=1}^{n} [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{34}} X_{3,i} | S^{N-n_{32}} X_{3,i})] \]
\[ \leq n(\max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+) \).

In (a), \(X_1^n\) in the second entropy term follows since given, \(M_{12}\) and \(X_2^n\), we may construct \(X_1^n\).
Fano as in deterministic Z channel

\[ n(R_{12} + R_{22} + R_{34} - \epsilon) \leq I(M_{12}; Y^n_2 | M_{21}, M_{23}, M_{43}) + I(M_{32}, M_{34}; Y^n_2, Y^n_4 | M_{43}, M_{12}, M_{21}, M_{23}) \]

\[ \leq H(Y^n_2 | M_{21}, M_{23}, M_{43}) + H(Y^n_4 | M_{43}, M_{12}, M_{21}, M_{23}, Y^n_2) \]

\[ \leq \sum_{i=1}^n [H(Y_{2,i} | Y_i^{i-1}, M_{21}, M_{23}, X^n_2) + H(Y_{4,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_i^{i-1}, X^n_4, Y^n_2, X^n_2)] \]

\[ \leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i})] \]

\[ + H(S^{N-n_{34}} X_{3,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_i^{i-1}, X^n_4, S^{N-n_{12}} X_{1,i} + S^{N-n_{22}} X_{2,i} + S^{N-n_{32}} X_{3,i}, X^n_2, X^n_1) \]

\[ \leq \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{34}} X_{3,i} | S^{N-n_{32}} X_{3,i})] \]

\[ \leq n(\max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+) \]

In (a), \( X^n_1 \) in the second entropy term follows since given, \( M_{12} \) and \( X^n_2 \), we may construct \( X^n_1 \).
Two-way linear deterministic IC

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{14}, n_{41}, n_{34}, n_{43}) \]

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4 \]

\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]

\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]

\[ Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]
Two-way linear deterministic IC

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{31}, n_{41}) \]

\[ Y_1 = S^{N-n_{11}}X_1 + S^{N-n_{21}}X_2 + S^{N-n_{12}}X_3 \]

\[ Y_2 = S^{N-n_{12}}X_1 + S^{N-n_{22}}X_2 + S^{N-n_{32}}X_3 \]

\[ Y_3 = S^{N-n_{23}}X_2 + S^{N-n_{33}}X_3 + S^{N-n_{23}}X_4 \]

\[ Y_4 = S^{N-n_{14}}X_1 + S^{N-n_{34}}X_3 + S^{N-n_{44}}X_4 \]

If partial adaptation:

\[ X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_{2i}^{-1}) \]

\[ X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_{4i}^{-1}) \]
Two-way linear deterministic IC

If partial adaptation:

Then capacity:

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{14}, n_{41}, n_{34}, n_{43}) \]
\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4 \]
\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]
\[ Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]

\[ X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_{2,i-1}) \]
\[ X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_{4,i-1}) \]

(A) IC in → direction

(B) IC in ← direction

\[ R_{12} \leq n_{12}, \quad R_{34} \leq n_{34}, \]
\[ R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \]
\[ R_{12} + R_{34} \leq \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+ \]
\[ R_{12} + R_{34} \leq \max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ 2R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{12} - n_{14}]^+ + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ R_{12} + 2R_{34} \leq \max(n_{34}, n_{14}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32}) \]

\[ R_{21} \leq n_{21}, \quad R_{43} \leq n_{43} \]
\[ R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+ \]
\[ R_{21} + R_{43} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ \]
\[ R_{21} + R_{43} \leq \max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ 2R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{21} - n_{23}]^+ + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ R_{21} + 2R_{43} \leq \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41}) \]
Two-way linear deterministic IC

If partial adaptation:

Then capacity:

\[ R_{12} \leq n_{12}, \quad R_{34} \leq n_{34}, \]
\[ R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \]
\[ R_{12} + R_{34} \leq \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+ \]
\[ R_{12} + R_{34} \leq \max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ 2R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{12} - n_{14}]^+ + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ R_{12} + 2R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32}) \]

(A) IC in \( \rightarrow \) direction

\[ R_{21} \leq n_{21}, \quad R_{43} \leq n_{43} \]
\[ R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+ \]
\[ R_{21} + R_{43} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ \]
\[ R_{21} + R_{43} \leq \max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ 2R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{21} - n_{23}]^+ + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ R_{21} + 2R_{43} \leq \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41}) \]

(B) IC in \( \leftarrow \) direction

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{14}, n_{41}, n_{34}, n_{43}) \]
\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4 \]
\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]
\[ Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]

\[ X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_2^{i-1}) \]
\[ X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_4^{i-1}) \]
Two-way linear deterministic IC

If partial adaptation:

Then capacity:

\[ R_{12} \leq n_{12}, \quad R_{34} \leq n_{34}, \]
\[ R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \]
\[ R_{12} + R_{34} \leq \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+ \]
\[ R_{12} + R_{34} \leq \max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ 2R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{12} - n_{14}]^+ + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ R_{12} + 2R_{34} \leq \max(n_{34}, n_{14}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32}) \]

\[ (A) \text{ IC in } \rightarrow \text{ direction} \]

\[ R_{21} \leq n_{21}, \quad R_{43} \leq n_{43} \]
\[ R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+ \]
\[ R_{21} + R_{43} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ \]
\[ R_{21} + R_{43} \leq \max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ 2R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{21} - n_{23}]^+ + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ R_{21} + 2R_{43} \leq \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41}) \]

\[ (B) \text{ IC in } \leftarrow \text{ direction} \]
Two-way linear deterministic IC

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{14}, n_{41}, n_{34}, n_{43}) \]

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4 \]

\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]

\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]

\[ Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]

If FULL adaptation:

\[ X_{1,i} = f_1(M_{12}, Y_1^{i-1}) \]
\[ X_{2,i} = f_1(M_{21}, Y_2^{i-1}) \]
\[ X_{3,i} = f_1(M_{34}, Y_3^{i-1}) \]
\[ X_{4,i} = f_1(M_{43}, Y_4^{i-1}) \]
Two-way linear deterministic IC

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{14}, n_{41}, n_{34}, n_{43}) \]

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4 \]
\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]
\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]
\[ Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]

If FULL adaptation:

Then we still have the outer bounds:

\[ R_{12} \leq n_{12}, \quad R_{34} \leq n_{34}, \]
\[ R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \]
\[ R_{12} + R_{34} \leq \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+ \]
\[ R_{12} + R_{34} \leq \max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ 2R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{12} - n_{14}]^+ + \max([n_{34} - n_{32}]^+, n_{14}) \]
\[ R_{12} + 2R_{34} \leq \max(n_{34}, n_{14}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32}) \]

(A) IC in \( \rightarrow \) direction

\[ R_{21} \leq n_{21}, \quad R_{43} \leq n_{43} \]
\[ R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+ \]
\[ R_{21} + R_{43} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ \]
\[ R_{21} + R_{43} \leq \max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ 2R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{21} - n_{23}]^+ + \max([n_{43} - n_{41}]^+, n_{23}) \]
\[ R_{21} + 2R_{43} \leq \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41}) \]

(B) IC in \( \leftarrow \) direction
Two-way linear deterministic IC

If FULL adaptation:

Then we still have the outer bounds:

(A) IC in → direction

\[ R_{12} \leq n_{12}, \quad R_{34} \leq n_{34}, \quad R_{12} \leq \max(n_{12}, n_{43}), \quad R_{34} \leq \max(n_{34}, n_{21}) \]

\[ R_{12} + R_{34} \leq \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+ \]

\[ R_{12} + R_{34} \leq \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+ \]

\[ R_{12} + 2R_{34} \leq \max(n_{34}, n_{14}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32}) \]

(B) IC in ← direction

\[ R_{21} \leq n_{21}, \quad R_{43} \leq n_{43}, \quad R_{21} \leq \max(n_{21}, n_{34}), \quad R_{43} \leq \max(n_{43}, n_{12}) \]

\[ R_{21} + R_{43} \leq \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+ \]

\[ R_{21} + R_{43} \leq \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+ \]

\[ R_{21} + 2R_{43} \leq \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41}) \]
Two-way linear deterministic IC

If FULL adaptation:

Then we still have the ETW-type bounds:

\[ N = \max(n_{11}, n_{22}, n_{33}, n_{44}, n_{12}, n_{21}, n_{32}, n_{23}, n_{14}, n_{41}, n_{34}, n_{43}) \]

\[ Y_1 = S^{N-n_{11}} X_1 + S^{N-n_{21}} X_2 + S^{N-n_{41}} X_4 \]

\[ Y_2 = S^{N-n_{12}} X_1 + S^{N-n_{22}} X_2 + S^{N-n_{32}} X_3 \]

\[ Y_3 = S^{N-n_{23}} X_2 + S^{N-n_{33}} X_3 + S^{N-n_{43}} X_4 \]

\[ Y_4 = S^{N-n_{14}} X_1 + S^{N-n_{34}} X_3 + S^{N-n_{44}} X_4 \]

\[ X_{1,i} = f(M_{12}, Y_i^{-1}) \]

\[ X_{2,i} = f(M_{21}, Y_2^{-1}) \]

\[ X_{4,i} = f(M_{43}, Y_4^{-1}) \]

\[ R_{12} \leq n_{12}, R_{34} \leq n_{34} \]

\[ 2R_{12} + R_{34} \leq \max(n_{34} - n_{32}^+, n_{14}) \]

\[ R_{12} + R_{34} \leq \max(n_{34} - n_{32}^+, n_{14}) + \max(n_{34} - n_{32}^+, n_{14}) \]

\[ R_{12} \leq \max(n_{34}, n_{14}) + \max(n_{34} - n_{32}^+, n_{14}) \]

\[ (A) \text{ IC in } \rightarrow \text{ direction} \]

\[ R_{21} \leq n_{21}, R_{43} \leq n_{43} \]

\[ R_{21} \leq \max(n_{21}, n_{34}), R_{43} \leq \max(n_{43}, n_{12}) \]

\[ 2R_{21} + R_{43} \leq \max(n_{21} - n_{23}^+, n_{43}) + \max(n_{43} - n_{41}^+, n_{23}) \]

\[ R_{21} + R_{43} \leq \max(n_{21} - n_{23}^+, n_{43}) + \max(n_{43} - n_{41}^+, n_{23}) \]

\[ R_{21} + R_{43} \leq \max(n_{21} - n_{23}^+, n_{43}) + \max(n_{43} - n_{41}^+, n_{23}) \]

\[ (B) \text{ IC in } \leftarrow \text{ direction} \]
Partial adaptation key lemma:

Lemma: Under partial adaptation conditions, for some deterministic functions $f_5$ and $f_6$,

$$X_{2,i} = f_5(M_{12}, M_{21}, M_{34}) \perp M_{43}, \forall i$$

$$X_{4,i} = f_6(M_{43}, M_{34}, M_{12}) \perp M_{21}, \forall i$$

where $\perp$ denotes independence.

Central to many of the converses!
Remark on partial adaptation:

\[ X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_{i-1}^2) \]

\[ X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_{i-1}^4) \]
Remark on partial adaptation:

$$X_1,i = f_1(M_{12}), \quad X_2,i = f_2(M_{21}, Y_{i-1}^2)$$

$$X_3,i = f_3(M_{34}), \quad X_4,i = f_4(M_{43}, Y_{i-1}^4)$$

**Blocks** routing at node 1,3
Remark on partial adaptation:

\[ X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_{i-1}^2) \]
\[ X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_{i-1}^4) \]

**Blocks** routing at node 1,3
Remark on partial adaptation:

\[ X_{1,i} = f_1(M_{12}) \]
\[ X_{2,i} = f_2(M_{21}, Y_{i-1}^2) \]
\[ X_{3,i} = f_3(M_{34}) \]
\[ X_{4,i} = f_4(M_{43}, Y_{i-1}^4) \]

**Blocks** routing at node 1,3

Adaptation useful in general!

Routing!
Remark on partial adaptation:

\[ X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_{i-1}^{2}) \]

\[ X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_{4}^{i-1}) \]

**Blocks** routing at node 1,3

**Adaptation useful in general!**

**Routing!**

However, it is sometimes useless!
Symmetric two-way linear deterministic IC
Symmetric two-way linear deterministic IC

Notice symmetry!
Symmetric two-way linear deterministic IC

Notice symmetry!
**Symmetric two-way linear deterministic IC**

Notice symmetry!

Define: \[ C_{sym}(\alpha) := \frac{R_{12} + R_{34}}{2} = \frac{R_{21} + R_{43}}{2}, \quad \alpha := \frac{n_I}{n_D} \]
Two-way interference channel related work

One-way IC

1 → 2: [El Gamal, Costa 1982]
3 → 4: [Bresler, Tse 2008]
1 → 3: [Etkin, Tse, Wang 2008]

2 messages
Two-way interference channel related work

One-way IC

[El Gamal, Costa 1982]

[Bresler, Tse 2008]

[Etkin, Tse, Wang 2008]

Two-way interference channel related work

One-way IC with FB

[Suh, Tse 2011]
Two-way interference channel related work

One-way IC

[El Gamal, Costa 1982]

[Bresler, Tse 2008]

[Etkin, Tse, Wang 2008]

One-way IC with rate-limited FB

[Vahid, Suh, Avestimehr 2012]

One-way IC with FB

[Suh, Tse 2011]
Two-way interference channel related work

One-way IC

- [El Gamal, Costa 1982]
- [Bresler, Tse 2008]
- [Etkin, Tse, Wang 2008]

Two-way interference channel

- Time-share forward + reverse

[1, 2, 3, 4]

One-way IC with rate-limited FB

[Vahid, Suh, Avestimehr 2012]

[1, 2, 3, 4]

One-way IC with interfering FB

[Suh, Wang, Tse 2012]

[1, 2, 3, 4]

One-way IC with FB

[Suh, Tse 2011]

[1, 2, 3, 4]
Two-way interference channel related work

One-way IC

1. [El Gamal, Costa 1982]
2. [Bresler, Tse 2008]
3. [Etkin, Tse, Wang 2008]

One-way IC with rate-limited FB

4. [Vahid, Suh, Avestimehr 2012]

One-way IC with interfering FB

5. [Suh, Wang, Tse 2012]

Time-share forward + reverse

Two-way IC

6. [Cheng, Devroye Allerton 2011]
7. [Cheng, Devroye CISS 2011]
8. [Cheng, Devroye ISIT 2012]
9. [Cheng, Devroye sub. Trans IT, 2012]
Symmetric sum-rate capacity comparison:

- One-way IC with perfect feedback
- One-way IC = Two-way IC with partial adaptation
- One-way IC with rate-limited feedback
- Two-way IC with full adaptation

This tells us that allowing partial adaptation is useless – i.e. may as well not adapt. Interestingly, the same holds true even for full adaptation for $\alpha > \frac{2}{3}$. This was also concluded for the linear deterministic one-way interference channel with interfering feedback links in [20]; what is interesting is that we can just as well squeeze in extra information messages in the feedback link—in the two-way interference channel model—rather than use the backwards links for feedback. The symmetric sum-capacity for the fully adaptive two-way IC remains open for $\alpha < \frac{2}{3}$; it is solved for partial adaptation.

Recently, the work in [23] has considered a one-way interference channel with interfering feedback links—again forming an interference channel—a generalization of some of the deterministic interference channels with feedback considered in [20] where the feedback link spends fraction $\lambda$ of its time sending feedback and uses the remaining $(1-\lambda)$ for other things—such as for example sending independent backwards messages—though adaptation as in [1] is not considered. This is quite different from our model which integrates sending feedback and messages over all links and does not force this separation. While the symmetric sum-capacity for this one-way interference channel with interfering feedback links is obtained in [23] in our notation for $\alpha \geq 1$ it is a function of this parameter $\lambda$ and is thus not plotted here.
Two-way capacity

≡

One-way \rightarrow

One-way \leftarrow

(as in two-way binary adder, two-way Gaussian channels)
Preview on Gaussian two-way interference channel

<table>
<thead>
<tr>
<th>Two-way Interference</th>
<th>Constant Gaps per user per direction, in bits (to outer bound)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Strong</td>
<td>0 (partial)</td>
</tr>
<tr>
<td>Strong</td>
<td>1 (full)</td>
</tr>
<tr>
<td>Weak INR &lt; 1</td>
<td>1 (full)</td>
</tr>
<tr>
<td>HK1 is active</td>
<td>1.5 (full)</td>
</tr>
<tr>
<td>HK2 is active</td>
<td></td>
</tr>
<tr>
<td>→ direction SNR ≤ INR^3</td>
<td>1 (partial)</td>
</tr>
<tr>
<td>← direction SNR &gt; INR^3</td>
<td>2 (partial)</td>
</tr>
</tbody>
</table>

**TABLE I**

*Constant gaps between non-adaptive symmetric Han and Kobayashi schemes in each direction and partially or fully adaptive outer bounds for the two-way Gaussian IC.*

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Conclusion

Adaptation appears to be useless when:

(a) Can cancel “self-interference”
(b) No coherent gains to be had
(c) No “routing” possible

[Cheng and Devroye, “Two-way Networks: when Adaptation is Useless”, Submitted to IEEE Trans. IT, available on arxiv.org.]

In general, when in adaptation useless (does not increase capacity region) in two-way networks?
Thank you!

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