An Outer Bound Region for the Parallel Two-way Channel with Interference

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***Abstract*—The classical interference channel models the communication limits of two independent, interfering streams of *one-way* data. In this paper we extend the classical interference channel model to a new channel model in which two streams of *two-way* data interfere with each other. In the absence of interference, this model would result in two parallel two-way channels (a four node channel); in the presence of interference it encompasses two-way, interference, and cooperation trade-offs. The discrete memoryless "parallel two-way channel with interference" is considered, in which each of the four nodes is the source of one message, the receiver of another, and experiences interference from yet another transmitter. The nodes may adapt their transmissions to the past received signals in a fully two-way fashion. We present an outer bound to the four dimensional capacity region which utilizes four auxiliary random variables to constrain the input distributions, and present a looser outer bound with a single auxiliary random variable which is computable as we place bounds on this variable's alphabet size.**

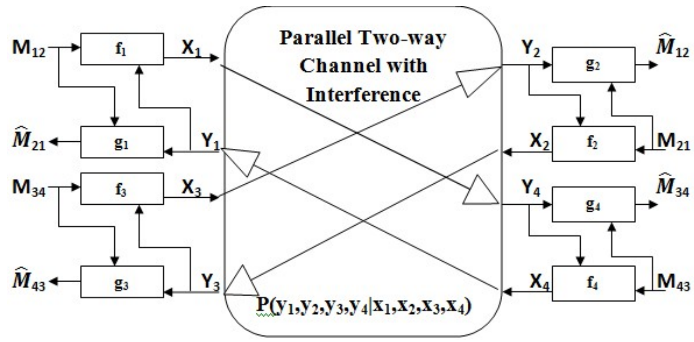

Fig. 1. Parallel Two-way Channel with Interference

I. Introduction

The two-way channel in which two nodes exchange a single two-way data stream (or two messages) over a common channel sheds light on the ability of nodes to adapt their transmissions to past received outputs. While the two-way channel has been extended to two-way relay channels in which a single two-way data stream is exchanged with the help of a relay, little work has considered multiple two-way data streams in a network setting. Two-way *networks* are particularly interesting not only because they extend one-way networks to allow for the more general class of two-way data, but they also emphasize the relationship between interference, two-way data – which amounts to nodes having additional own-message side-information as well as being able to adapt their transmissions to received data – and cooperation in a network setting.

Towards this goal, we start off by considering the simplest relay-free non-trivial network which contains multiple two-way data streams that interfere. We consider two parallel two-way channels in which each node's transmission is heard by its desired receiver as well as one undesired receiver.

A. Past Work

This channel model merges elements of the two-way, interference, and causally cooperative (or forms of generalized feedback) channel models, but has not been explicitly considered before. We consider the general class of discrete memoryless two-way channels first introduced by Shannon [9] who proposed inner and outer bounds of the same form/expression, but taken over independent and fully general input distributions respectively. We will use outer-bounding techniques similar to those used to improve Shannon's outer bound, as first done by [12] for the two-way channel, and later also used in deriving outer bounds for a two-way multiple-access broadcast-channel model considered in [3]. This last channel is one of the few two-way networks considered in the literature and is most related to the channel considered here. [3] derives outer bounds on the capacity region of a three node network where the two terminal nodes each have a message for the base-station and the base-station has a common message for the terminal nodes. The nodes are half-duplex and may adapt their current transmissions to previously received signals, as in general (non-restricted) two-way channels. This paper's channel model differs as it considers four full-duplex nodes with four messages and captures interference effects (two interference channels in the $\rightarrow$ and $\leftarrow$ directions). Numerous interference channel outer bounds have been derived, including those for Gaussian channels [4], [7], discrete memoryless channels [5] with strong interference conditions [8], with feedback [10], and with generalized feedback [11].

B. Outline of Paper

We define the discrete memoryless, full-duplex, parallel two-way channel with interference channel model in Section II, derive two outer bounds for this channel in Section III – one tighter but a function of four auxiliary random variables and one looser but computable due to the cardinality bound on the single auxiliary random variable, compare the tighter bound to related models in Section IV, and conclude in Section V.

II. Channel Model

We consider a multi-user network which we term the parallel two-way channel with interference (PTW-IF), which is shown in Fig.1. This network has four distributed transmitters (encoders) and four distributed receivers (decoders), which encode/decode four independent messages. Transmitter 1 and 3 send messages M_{12} and M_{34} to receiver 2 and 4 respectively. Similarly, transmitter 2 and 4 send messages M_{21} and M_{43} to receiver 1 and 3 respectively. Because of the two-way feature with interference, the outputs of receiver 1 and 3 depend (possibly in a noisy fashion) on the inputs at transmitter 1,2,4 and 2,3,4 respectively. Similarly, the outputs of receiver 2 and 4 depend on the inputs at transmitter 1,2,3 and 1,3,4 respectively. We note that a more general channel model would allow for the output at each node to depend on all four inputs, but that we consider this somewhat simplified model at first given its already challenging nature, and expect to be able to carefully generalize these results to the general parallel two-way interference channel (rather than parallel two-way channel with interference).

The channel input and output of user $i \in \{1,2,3,4\}$ at discrete time k are $X_{i,k}$ and $Y_{i,k}$, which lie in the alphabets $\mathcal{X}_i$ and $\mathcal{Y}_i$ respectively. The channel is discrete and memoryless and is thus fully described by the transition probability matrix $\{p(y_1,y_2,y_3,y_4|x_1,x_2,x_3,x_4)\}$ which we assume has the property that:

$$\begin{aligned} Y_1 &= T_1(X_1,X_2,X_4) \\ Y_2 &= T_2(X_1,X_2,X_3) \\ Y_3 &= T_3(X_2,X_3,X_4) \\ Y_4 &= T_4(X_1,X_3,X_4), \end{aligned}$$

where T_l ($l \in \{1,2,3,4\}$) are discrete memoryless mappings which may include an element of randomness (or noise). By introducing the notation $A_i^j = (A_{i,1}, A_{i,2}, ..., A_{i,j})$, for any given time j, we may describe the encoding functions which yield the channel inputs at time j as follows:

$$\begin{aligned} X_{1,j} &= f_1(M_{12}, Y_1^{j-1}) \\ X_{2,j} &= f_2(M_{21}, Y_2^{j-1}) \\ X_{3,j} &= f_3(M_{34}, Y_3^{j-1}) \\ X_{4,j} &= f_4(M_{43}, Y_4^{j-1}), \end{aligned}$$

where f_i ($i \in \{1,2,3,4\}$) are deterministic functions. For a given blocklength n, at time step $0 \leq j \leq n$, the encoder selects the next input $X_{i,j}(M_{ik}, Y_i^{j-1})$ out of the $2^{nR_{ik}}$ ($R_{ik} \geq 0$ is the rate of communication between sender i and receiver k) codewords $X_i^n(M_{ik}, Y_i^{n-1})$. Receiver k uses a decoding function $g_k : \mathcal{Y}_k^n \to \widehat{\mathcal{M}}_{ik}$ to obtain an estimate $\widehat{M}_{ik}$ of the transmitted message M_{ik}. Standard definitions for achievable rate regions and capacity regions for the rates R_{ik} are used [1]. We seek outer bounds to the four-dimensional $(R_{12}, R_{21}, R_{34}, R_{43})$ capacity region.

III. Outer Bounds Region for PTW-IF

We now present an outer bound to the capacity of this channel; remarks and interpretations of this bound are found at the end of this section.

Theorem 1: The capacity region $\mathcal{C}$ of the parallel two-way channel with interference is a subset of the region $\mathcal{C}^*$:

$$\begin{aligned}
&\mathcal{C}^* \equiv \{(R_{12}, R_{21}, R_{34}, R_{43}) : \\
&R_{12} \leq \min\{I(X_1,Z_1;Y_2|X_2,Z_2,M_{34},M_{43},Q), H(X_1|Z_1,,Z_2,X_2,Q), \\
&\quad I(X_2,X_4;Y_3|X_3,Z_3,Q) + I(X_1;Y_2|X_2,X_3,Z_2,Z_3,Q)\} \quad (1)\\
&R_{21} \leq \min\{I(X_2,Z_2;Y_1|X_1,Z_1,M_{34},M_{43},Q), H(X_2|Z_1,Z_2,X_1,Q), \\
&\quad I(X_1,X_3;Y_4|X_4,Z_4,Q) + I(X_2;Y_1|X_1,X_4,Z_1,Z_4,Q)\} \quad (2)\\
&R_{34} \leq \min\{I(X_3,Z_3;Y_4|X_4,Z_4,M_{12},M_{21},Q), H(X_3|Z_3,Z_4,X_4,Q), \\
&\quad I(X_2,X_4;Y_1|X_1,Z_1,Q) + I(X_3;Y_4|X_4,X_1,Z_4,Z_1,Q)\} \quad (3)\\
&R_{43} \leq \min\{I(X_4,Z_4;Y_3|X_3,Z_3,M_{12},M_{21},Q), H(X_4|Z_3,Z_4,X_3,Q), \\
&\quad I(X_1,X_3;Y_2|X_2,Z_2,Q) + I(X_4;Y_3|X_3,X_2,Z_3,Z_2,Q)\} \quad (4)\\
&R_{12}+R_{34} \leq I(X_1,X_3;Y_2,Y_4|X_2,X_4,Z_2,Z_4,Q) \quad (5)\\
&R_{12}+R_{43} \leq I(X_1,X_4;Y_2,Y_3|X_2,X_3,Z_2,Z_3,Q) \quad (6)\\
&R_{21}+R_{34} \leq I(X_2,X_3;Y_1,Y_4|X_1,X_4,Z_1,Z_4,Q) \quad (7)\\
&R_{21}+R_{43} \leq I(X_2,X_4;Y_1,Y_3|X_1,X_3,Z_1,Z_3,Q) \quad (8)\\
&R_{12}+R_{21}+R_{34} \leq I(X_1,X_2,X_3;Y_1,Y_2,Y_4|X_4,Z_4,Q) \quad (9)\\
&R_{12}+R_{21}+R_{43} \leq I(X_1,X_2,X_4;Y_1,Y_2,Y_3|X_3,Z_3,Q) \quad (10)\\
&R_{12}+R_{43}+R_{34} \leq I(X_1,X_4,X_3;Y_3,Y_2,Y_4|X_2,Z_2,Q) \quad (11)\\
&R_{21}+R_{43}+R_{34} \leq I(X_4,X_2,X_3;Y_1,Y_3,Y_4|X_1,Z_1,Q)\} \quad (12)
\end{aligned}$$

where $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Z_1, Z_2, Z_3, Z_4, Q$ are random variables subject to the following input distribution:

$$p(z_1,z_2,z_3,z_4|q)p(x_1|z_1,q)p(x_2|z_2,q)p(x_3|z_3,q)p(x_4|z_4,q). \quad (13)$$

Proof: By symmetry, we need only demonstrate bounds (1), (5), (9). For simplicity, we omit the standard time-sharing random variable arguments for Q and refer the reader to standard arguments as in [1].

Proof of bound (1)*:* We obtain three bounds on the single message rates; the first two are in the spirit of [12] for the two-way channel, while the third is reminiscent of the two-way multiple-access / broadcast channel bounds of [3].

$$\begin{aligned}
&nR_{12} = H(M_{12}|M_{21},M_{34},M_{43}) \\
&= H(M_{12}|M_{21},M_{34},M_{43},Y_2) + I(M_{12};Y_2|M_{34},M_{43},M_{21}) \\
&\overset{(a)}{\leq} n\epsilon + I(M_{12};Y_2|M_{34},M_{43},M_{21}) \\
&= H(Y_2|M_{34},M_{43},M_{21}) - H(Y_2|M_{12},M_{21},M_{34},M_{43}) + n\epsilon \\
&= \sum_{j=1}^{n} [H(Y_{2,j}|M_{34},M_{43},M_{21},Y_2^{j-1}) \\
&\quad - H(Y_{2,j}|M_{12},M_{21},M_{34},M_{43},Y_2^{j-1})] + n\epsilon
\end{aligned}$$

$$\overset{(b)}{\leq} \sum_{j=1}^{n} [H(Y_{2,j}|M_{34}, M_{43}, M_{21}, Y_2^{j-1}, X_2^j)$$
$$- H(Y_{2,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_2^{j-1}, X_2^j, Y_1^{j-1})] + n\epsilon$$
$$\overset{(c)}{=} \sum_{j=1}^{n} [H(Y_{2,j}|M_{34}, M_{43}, M_{21}, Y_2^{j-1}, X_2^j, Z_{2,j})$$
$$- H(Y_{2,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_1^{j-1} Y_2^{j-1}, X_2^j, X_1^j, Z_{1,j}, Z_{2,j})] + n\epsilon$$
$$\overset{(d)}{\leq} \sum_{j=1}^{n} [H(Y_{2,j}|X_{2,j}, Z_{2,j}, M_{34}, M_{43}) - H(Y_{2,j}|M_{34}, M_{43}, X_{2,j}, X_{1,j}, Z_{1,j}, Z_{2,j})] + n\epsilon$$
$$= \sum_{j=1}^{n} I(X_{1,j}, Z_{1,j}; Y_{2,j}|X_{2,j}, Z_{2,j}, M_{34}, M_{43}) + n\epsilon$$

where (a) follows from Fano's inequality, (b) uses $X_2^j = f_2(M_{21}, Y_2^{j-1})$ and conditioning reduces entropy when a "genie" provides Y_1^{j-1} in the negative term. We introduce new random variables $Z_{i,j} = (X_i^{j-1}, Y_i^{j-1})$ in (c), and also use $X_1^j = f_1(M_{12}, Y_1^{j-1})$. In (d), the first term follows as conditioning reduces entropy; the second term, follows since given $M_{34}, M_{43}, X_{1,j}, X_{2,j}, Z_{1,j}, Z_{2,j}$, the channel output $Y_{2,j}$ is independent of the other terms. This is illustrated by the Markov chain diagram in Fig. 2, where we see that due to the dependence of Y_{2j} on M_{34} and M_{43} (since neither $X_{3,j}$ nor $Y_{3,j}$ are given) these terms may not be dropped. By introducing a time sharing random variable Q and using arguments as in [1], we obtain $R_{12} \leq I(X_1, Z_1; Y_2|X_2, Z_2, M_{34}, M_{43}, Q)$. One problem with this bound is the presence of the messages. We now derive two other bounds on R_{12} in which this

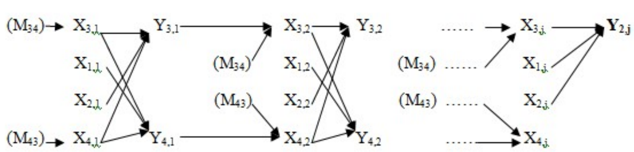

Fig. 2. First example of the Markov chain used in Theorem 1.

dependence on the messages is removed. To do so, notice

$$I(M_{12}; Y_2|M_{34}, M_{43}, M_{21})$$
$$\leq I(M_{12}; Y_2, Y_1, X_1|M_{21}, M_{34}, M_{43})$$
$$= H(Y_1, Y_2, X_1|M_{21}, M_{34}, M_{43})$$
$$- H(Y_1, Y_2, X_1|M_{12}, M_{21}, M_{34}, M_{43})$$
$$= \sum_{j=1}^{n} [H(Y_{1,j}, Y_{2,j}, X_{1,j}|M_{21}, M_{34}, M_{43}, X_1^{j-1}, Y_1^{j-1}, Y_2^{j-1})$$
$$- H(Y_{1,j}, Y_{2,j}, X_{1,j}|M_{12}, M_{21}, M_{34}, M_{43}, X_1^{j-1}, Y_1^{j-1}, Y_2^{j-1})]$$
$$\overset{(a)}{=} \sum_{j=1}^{n} [H(X_{1,j}|M_{21}, M_{34}, M_{43}, X_1^{j-1}, Y_1^{j-1}, Y_2^{j-1}, X_2^j)$$

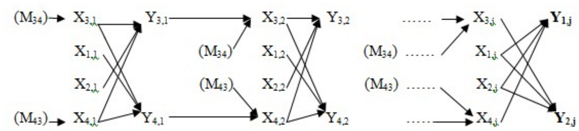

Fig. 3. Second example of the Markov chain used in Theorem 1.

$$+ H(Y_{1,j}, Y_{2,j}|X_{1,j}, M_{21}, M_{34}, M_{43}, X_1^{j-1}, Y_1^{j-1}, Y_2^{j-1}, X_2^j)$$
$$- H(X_{1,j}|M_{12}, M_{21}, M_{34}, M_{43}, X_1^{j-1}, Y_1^{j-1}, Y_2^{j-1}, X_2^j)$$
$$- H(Y_{1,j}, Y_{2,j}|X_{1,j}, M_{12}, M_{21}, M_{34}, M_{43}, X_1^{j-1}, Y_1^{j-1}, Y_2^{j-1}, X_2^j)]$$
$$\overset{(b)}{\leq} \sum_{j=1}^{n} [H(X_{1,j}|Z_{1,j}, Z_{2,j}, X_{2,j})]$$

where (a) uses the chain rule and $X_2^j = f_2(M_{21}, Y_2^{j-1})$, and (b) follows from: 1) the third term in (a) is zero since $X_{1,j} = f_1(M_{12}, Y_1^{j-1})$. 2) The second term and the fourth term in (a) cancel each other by the Markov chain given in Fig. 3, where we see that given $X_{1,j}, X_1^{j-1}, X_2^j, M_{34}, M_{43}$, the channel outputs $Y_{1,j}$ and $Y_{2,j}$ are independent of anything else that is given. 3) For the first term in (a), we introduce a new random variable $Z_{1,j} = (X_1^{j-1}, Y_1^{j-1})$, and the inequality follows as conditioning reduces entropy. Again, by introducing a time sharing random variable Q and using arguments as in [1], we obtain $R_{12} \leq H(X_1|Z_1, Z_2, X_2, Q)$.

In the previous bound, X_1 and Y_1 were provided as genie-aided side-information at node 2. Now we derive another outer bound on R_{12} by giving Y_3 as genie-aided side-information at node 2. That is,

$$I(M_{12}; Y_2|M_{34}, M_{43}, M_{21})$$
$$\leq I(M_{12}; Y_2, Y_3|M_{34}, M_{43}, M_{21})$$
$$= I(M_{12}; Y_3|M_{34}, M_{43}, M_{21})$$
$$+ I(M_{12}; Y_2|M_{34}, M_{43}, M_{21}, Y_3) \quad (14)$$

Now we upper bound the first term in (14) as follows:

$$I(M_{12}; Y_3|M_{34}, M_{43}, M_{21})$$
$$= H(Y_3|M_{34}, M_{43}, M_{21}) - H(Y_3|M_{12}, M_{21}, M_{34}, M_{43})$$
$$= \sum_{j=1}^{n} [H(Y_{3,j}|M_{34}, M_{43}, M_{21}, Y_3^{j-1})$$
$$- H(Y_{3,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_3^{j-1})]$$
$$\overset{(a)}{\leq} \sum_{j=1}^{n} [H(Y_{3,j}|M_{34}, M_{43}, M_{21}, Y_3^{j-1}, X_3^j)$$
$$- H(Y_{3,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_3^{j-1}, X_3^j, X_4^j, X_2^j)]$$
$$\overset{(b)}{=} \sum_{j=1}^{n} [H(Y_{3,j}|M_{34}, M_{43}, M_{21}, Y_3^{j-1}, X_3^j, Z_{3,j})$$
$$- H(Y_{3,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_3^{j-1}, X_3^j, Z_{3,j}, X_4^j, X_2^j)]$$

$$\overset{(c)}{\leq} \sum_{j=1}^{n} [H(Y_{3,j}|X_{3,j}, Z_{3,j}) - H(Y_{3,j}|X_{2,j}, X_{3,j}, X_{4,j}, Z_{3,j})]$$

$$= \sum_{j=1}^{n} I(X_{2,j}, X_{4,j}; Y_{3,j}|X_{3,j}, Z_{3,j}) \quad (15)$$

where (a) uses $X_3^j = f_3(M_{34}, Y_3^{j-1})$, and the inequality holds as we added X_2^j, X_4^j in the negative term (thereby reducing it). In (b) we introduce a new random variable $Z_{3,j} = (X_3^{j-1}, Y_3^{j-1})$. In (c), for the first term, conditioning reduces entropy. For the second term, we again use the Markov chain properties similar to those in Figs. 2 and 3 to see that given $X_{2,j}, X_{3,j}, X_{4,j}$, the channel output $Y_{3,j}$ is independent of all other terms. Now we proceed to upper bound the second term in (14):

$$\begin{aligned}
&I(M_{12}; Y_2|M_{34}, M_{43}, M_{21}, Y_3) \\
&= H(Y_2|M_{34}, M_{43}, M_{21}, Y_3) \\
&\qquad - H(Y_2|M_{12}, M_{21}, M_{34}, M_{43}, Y_3) \\
&= \sum_{j=1}^{n} [H(Y_{2,j}|M_{34}, M_{43}, M_{21}, Y_2^{j-1}, Y_3) \\
&\quad - H(Y_{2,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_2^{j-1}, Y_3)] \\
&\overset{(d)}{\leq} \sum_{j=1}^{n} [H(Y_{2,j}|M_{34}, M_{43}, M_{21}, Y_2^{j-1}, X_2^j, Y_3^{j-1}, X_3^j) \\
&\quad - H(Y_{2,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_2^{j-1}, X_2^j, X_3^j, X_1^j, Y_3)] \\
&\overset{(e)}{=} \sum_{j=1}^{n} [H(Y_{2,j}|M_{34}, M_{43}, M_{21}, Y_2^{j-1}, X_2^j, Y_3^{j-1}, X_3^j, \\
&\quad Z_{2,j}, Z_{3,j}) - H(Y_{2,j}|M_{12}, M_{21}, M_{34}, M_{43}, \\
&\quad Y_2^{j-1}, X_2^j, X_3^j, X_1^j, Y_3, Z_{2,j}, Z_{3,j})] \\
&\overset{(f)}{\leq} \sum_{j=1}^{n} [H(Y_{2,j}|X_{2,j}, X_{3,j}, Z_{2,j}, Z_{3,j}) \\
&\quad - H(Y_{2,j}|X_{3,j}, X_{2,j}, X_{1,j}, Z_{3,j}, Z_{2,j})] \\
&= \sum_{j=1}^{n} I(X_{1,j}; Y_{2,j}|X_{2,j}, X_{3,j}, Z_{2,j}, Z_{3,j}) \qquad (16)
\end{aligned}$$

where (d) uses $X_2^j = f_2(M_{21}, Y_2^{j-1})$ and $X_3^j = f_3(M_{34}, Y_3^{j-1})$, and the addition of X_1^j in the negative term which leads to the inequality. In (e), we introduce new random variables $Z_{i,j} = (X_i^{j-1}, Y_i^{j-1})$. In (f), for the first term, conditioning reduces entropy. For the second term, we again use the iterated Markov chain as in Fig. 1 to see that given $X_{1,j}, X_{2,j}, X_{3,j}$, the channel output $Y_{2,j}$ is independent of all other terms. By combining (15) and (16), and introducing a time sharing random variable Q and using arguments as in [1], we obtain $R_{12} \leq I(X_2, X_4; Y_3|X_3, Z_3) + I(X_1; Y_2|X_2, X_3, Z_2, Z_3)$.

Proof of bound (5)*:*

$$\begin{aligned}
&n(R_{12} + R_{34}) = H(M_{12}, M_{34}|M_{21}, M_{43}) \\
&= H(M_{12}, M_{34}|M_{21}, M_{43}, Y_2, Y_4) \\
&\quad + I(M_{12}, M_{34}; Y_2, Y_4|M_{43}, M_{21}) \\
&\overset{(a)}{\leq} n\epsilon + I(M_{12}, M_{34}; Y_2, Y_4|M_{43}, M_{21}) \\
&= H(Y_2, Y_4|M_{43}, M_{21}) \\
&\quad - H(Y_2, Y_4|M_{12}, M_{21}, M_{34}, M_{43}) + n\epsilon \\
&= \sum_{j=1}^{n} [H(Y_{2,j}, Y_{4,j}|M_{43}, M_{21}, Y_2^{j-1}, Y_4^{j-1}) \\
&\quad - H(Y_{2,j}, Y_{4,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_2^{j-1}, Y_4^{j-1})] + n\epsilon \\
&\overset{(b)}{\leq} \sum_{j=1}^{n} [H(Y_{2,j}, Y_{4,j}|M_{43}, M_{21}, Y_2^{j-1}, Y_4^{j-1}, X_2^j, X_4^j) \\
&\quad - H(Y_{2,j}, Y_{4,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_2^{j-1}, Y_4^{j-1}, \\
&\qquad X_2^j, X_4^j, X_1^j, X_3^j)] + n\epsilon \\
&\overset{(c)}{=} \sum_{j=1}^{n} [H(Y_{2,j}, Y_{4,j}|M_{43}, M_{21}, Y_2^{j-1}, Y_4^{j-1}, X_2^j, X_4^j, \\
&\quad Z_{2,j}, Z_{4,j}) - H(Y_{2,j}, Y_{4,j}|M_{12}, M_{21}, M_{34}, M_{43}, \\
&\quad Y_2^{j-1}, Y_4^{j-1}, X_2^j, X_4^j, X_1^j, X_3^j, Z_{2,j}, Z_{4,j})] + n\epsilon \\
&\overset{(d)}{\leq} \sum_{j=1}^{n} [H(Y_{2,j}, Y_{4,j}|X_{2,j}, X_{4,j}, Z_{2,j}, Z_{4,j}) \\
&\quad - H(Y_{2,j}, Y_{4,j}|X_{2,j}, X_{4,j}, X_{1,j}, X_{3,j}, Z_{2,j}, Z_{4,j})] + n\epsilon \\
&= \sum_{j=1}^{n} I(X_{1,j}, X_{3,j}; Y_{2,j}, Y_{4,j}|X_{2,j}, X_{4,j}, Z_{2,j}, Z_{4,j}) + n\epsilon
\end{aligned}$$

where (a) follows from Fano's inequality, (b) uses $X_2^j = f_2(M_{21}, Y_2^{j-1})$ and $X_4^j = f_4(M_{43}, Y_4^{j-1})$, and the inequality follows from giving the terms X_1^j, X_3^j as genie-aided information in the negative term. We introduce new random variables $Z_{i,j} = (X_i^{j-1}, Y_i^{j-1})$ in (c). In (d), for the first term, conditioning reduces entropy. For the second term, we use Markov chain arguments similar to those in Figs. 2 and 3 to see that given $X_{1,j}, X_{2,j}, X_{3,j}, X_{4,j}$, the channel outputs $Y_{2,j}, Y_{4,j}$ are independent of anything else that is given. By introducing a time sharing random variable Q and using arguments as in [1], we obtain $R_{12} + R_{34} \leq I(X_1, X_3; Y_2, Y_4|X_2, X_4, Z_2, Z_4, Q)$.

Proof of bound (9):

$$\begin{aligned}
&n(R_{12} + R_{21} + R_{34}) = H(M_{12}, M_{21}, M_{34}|M_{43}) \\
&= H(M_{12}, M_{21}, M_{34}|M_{43}, Y_1, Y_2, Y_4) \\
&\quad + I(M_{12}, M_{21}, M_{34}; Y_1, Y_2, Y_4|M_{43}) \\
&\overset{(a)}{\leq} n\epsilon + I(M_{12}, M_{21}, M_{34}; Y_1, Y_2, Y_4|M_{43}) \\
&= H(Y_1, Y_2, Y_4|M_{43}) \\
&\quad - H(Y_1, Y_2, Y_4|M_{12}, M_{21}, M_{34}, M_{43}) + n\epsilon
\end{aligned}$$

$$= \sum_{j=1}^{n}[H(Y_{1,j}, Y_{2,j}, Y_{4,j}|M_{43}, Y_1^{j-1}, Y_2^{j-1}, Y_4^{j-1}) - H(Y_{1,j}, Y_{2,j}, Y_{4,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_1^{j-1}, Y_2^{j-1}, Y_4^{j-1})] + n\epsilon$$

$$\overset{(b)}{\le} \sum_{j=1}^{n}[H(Y_{1,j}, Y_{2,j}, Y_{4,j}|M_{43}, Y_1^{j-1}, Y_2^{j-1}, Y_4^{j-1}, X_4^j) - H(Y_{1,j}, Y_{2,j}, Y_{4,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_1^{j-1}, Y_2^{j-1}, Y_4^{j-1}, X_1^j, X_2^j, X_4^j, X_3^j)] + n\epsilon$$

$$\overset{(c)}{=} \sum_{j=1}^{n}[H(Y_{1,j}, Y_{2,j}, Y_{4,j}|M_{43}, Y_1^{j-1}, Y_2^{j-1}, Y_4^{j-1}, X_4^j, Z_{4,j}) - H(Y_{1,j}, Y_{2,j}, Y_{4,j}|M_{12}, M_{21}, M_{34}, M_{43}, Y_1^{j-1}, Y_2^{j-1}, Y_4^{j-1}, X_1^j, X_2^j, X_4^j, X_3^j, Z_{4,j})] + n\epsilon$$

$$\overset{(d)}{\le} \sum_{j=1}^{n}[H(Y_{1,j}, Y_{2,j}, Y_{4,j}|X_{4,j}, Z_{4,j}) - H(Y_{1,j}, Y_{2,j}, Y_{4,j}|X_{2,j}, X_{4,j}, X_{1,j}, X_{3,j}, Z_{4,j})] + n\epsilon$$

$$= \sum_{j=1}^{n} I(X_{1,j}, X_{2,j}, X_{3,j}; Y_{1,j}, Y_{2,j}, Y_{4,j}|X_{4,j}, Z_{4,j}) + n\epsilon$$

where (a) follows from Fano's inequality, (b) uses $X_4^j = f_4(M_{43}, Y_4^{j-1})$, $X_1^j = f_1(M_{12}, Y_1^{j-1})$ and $X_2^j = f_2(M_{21}, Y_2^{j-1})$ to include X_4^j, X_1^j and X_2^j for free. The inequality follows by giving the genie-aided information X_3^j in the negative term. We introduce a new random variable $Z_{4,j} = (X_4^{j-1}, Y_4^{j-1})$ in (c). In (d), for the first term, conditioning reduces entropy. For the second term, we again use a Markov chain argument similar to those in Figs. 2 and 3 to see that given $X_{1,j}, X_{2,j}, X_{3,j}, X_{4,j}$, the channel outputs $Y_{1,j}, Y_{2,j}, Y_{4,j}$ are independent of anything else that is given. By introducing a time sharing random variable Q and using arguments as in [1], we obtain $R_{12} + R_{21} + R_{34} \le I(X_1, X_2, X_3; Y_1, Y_2, Y_4|X_4, Z_4, Q)$.

That the channel input distribution splits according to (13) is omitted due to space, but follows along the lines of [12]. ■

Remark 2: The derivation of the bound closely follows the ideas of [12] for the point-to-point two-way channel as well as those of [3] for the two-way multiple-access and broadcast channel with a common message. The main difference is the structure of our channel, which contains interference but no multiple-access or broadcast elements. In addition, our bounds are derived in full-duplex scenario, which is different from [3]'s half-duplex model.

Remark 3: The third term in bound (1) suggests interpreting users 3 and 4 as relay nodes for the transmission from user 1 to receiver 2. Indeed, in the PTW-IF, the message M_{12} may be transmitted from $1 \to 2$ directly, or may intuitively be "routed" through $1 \to 4 \to 3 \to 2$, in a cooperative fashion. This ability to cooperate is captured by the third bound in (1).

Remark 4: We note that the double and triple rate bounds (5) – (12) are quite intuitive and follow in a relatively straightforward manner; the key potential improvement over other cut-set like outer bounds is the somewhat more restrained input distribution due to the auxiliary Z random variables over which this bound is taken.

We now give constraints on the sizes of the alphabets of the auxiliary random variables $Z_i, i \in (1, 2, 3, 4)$ in the following theorem, which presents a potentially weaker but computable (bounded auxiliary random variable sizes) outer bound region.

Theorem 5: The capacity region $\mathcal{C}$ of the PTW-IF is a subset of the region $\mathcal{C}^{**}$:

$$\mathcal{C}^{**} := \{(R_{12}, R_{21}, R_{34}, R_{43}) :$$

$$R_{12} \le \min\{H(X_1|X_2, X, Q), I(X_2, X_4; Y_3|X_3, Q) + I(X_1; Y_2|X_2, X_3, Q)\} \quad (17)$$

$$R_{21} \le \min\{H(X_2|X_1, X, Q), I(X_1, X_3; Y_4|X_4, Q) + I(X_2; Y_1|X_1, X_4, Q)\} \quad (18)$$

$$R_{34} \le \min\{H(X_3|X_4, X, Q), I(X_2, X_4; Y_1|X_1, Q) + I(X_3; Y_4|X_1, X_4, Q)\} \quad (19)$$

$$R_{43} \le \min\{H(X_4|X_3, X, Q), I(X_3, X_1; Y_2|X_2, Q) + I(X_4; Y_3|X_2, X_3, Q)\} \quad (20)$$

$$R_{12} + R_{34} \le I(X_1, X_3; Y_2, Y_4|X_2, X_4, Q) \quad (21)$$

$$R_{12} + R_{43} \le I(X_1, X_4; Y_2, Y_3|X_2, X_3, Q) \quad (22)$$

$$R_{21} + R_{34} \le I(X_2, X_3; Y_1, Y_4|X_1, X_4, Q) \quad (23)$$

$$R_{21} + R_{43} \le I(X_2, X_4; Y_1, Y_3|X_1, X_3, Q) \quad (24)$$

$$R_{12} + R_{21} + R_{34} \le I(X_1, X_2, X_3; Y_1, Y_2, Y_4|X_4, Q) \quad (25)$$

$$R_{12} + R_{21} + R_{43} \le I(X_1, X_2, X_4; Y_1, Y_2, Y_3|X_3, Q) \quad (26)$$

$$R_{12} + R_{43} + R_{34} \le I(X_1, X_4, X_3; Y_3, Y_2, Y_4|X_2, Q) \quad (27)$$

$$R_{21} + R_{43} + R_{34} \le I(X_2, X_3, X_4; Y_1, Y_3, Y_4|X_1, Q) \quad (28)$$

where $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, X, Q$ are random variables subject to the following input distribution:

$$p(x|q)p(x_1|x, q)p(x_2|x, q)p(x_3|x, q)p(x_4|x, q),$$

subject to the cardinality bound $|\mathcal{X}| \le |\mathcal{X}_1||\mathcal{X}_2||\mathcal{X}_3||\mathcal{X}_4| + 3$.

Proof: In Theorem 1, let $X = (Z_1, Z_2, Z_3, Z_4)$. Then bound (1) in Theorem 1 may be relaxed to

$$R_{12} \le \min\{H(X_1|X_2, X, Q), H(Y_3|X_3, Q) - H(Y_3|X_2, X_3, X_4, Q) + H(Y_2|X_2, X_3, Q) - H(Y_2|X_1, X_2, X_3, Q)\}$$
$$= \min\{H(X_1|X_2, X, Q), I(X_2, X_4; Y_3|X_3, Q) + I(X_1; Y_2|X_2, X_3, Q)\}$$

which is bound (17) in Theorem 5. Note that $H(X_1|X_2, Z_1, Z_2) = H(X_1|X_2, Z_1, Z_2, Z_3, Z_4)$ due to the form of the joint distribution in Theorem 1. All other bounds in Theorem 5 may similarly be obtained from Theorem 1. We note that we have further loosened bounds by dropping all bounds which depended on the messages. According to the support lemma of [2, p.310], the cardinality bound may be proven as follows: We can find a subset of $\mathcal{X}$, say $\mathcal{X}^{'}$, with the constraint $|\mathcal{X}^{'}| \le |\mathcal{X}_1||\mathcal{X}_2||\mathcal{X}_3||\mathcal{X}_4| + 3$, and let $p(x^{'})$ be the distribution corresponding to $\mathcal{X}^{'}$. Now, if the input distribution is of the form $p(x^{'})p(x_1|x)p(x_2|x)p(x_3|x)p(x_4|x)$, then equating $\{p(x_1, x_2, x_3, x_4) : x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2, x_3 \in \mathcal{X}_3, x_4 \in \mathcal{X}_4\}$, and all the entropy terms (there are four) in the bounds of Theorem 5 with the new x' variables ensures that all the mutual information terms also remain unchanged. Therefore, we claim that the cardinality bound is $|\mathcal{X}| \leq (|\mathcal{X}_1||\mathcal{X}_2||\mathcal{X}_3||\mathcal{X}_4| - 1) + 4 = |\mathcal{X}_1||\mathcal{X}_2||\mathcal{X}_3||\mathcal{X}_4| + 3$. ■

IV. Discussion

The PTW-IF is a fairly general channel model which combines elements of two-way channels with those of interference channels. As such, we expect an outer bound for this channel to relate to outer bounds for similar channel models and briefly discuss this next.

A. The cut-set outer bound for the PTW-IF

A readily available outer bound to this channel model (and may be added to those of Theorem 1 to further potentially tighten the outer bound) is the standard cut-set outer bound [1], [6] , which, for the PTW-IF, is given by the region:

$$R_{12} \leq I(X_1; Y_2, Y_4|X_2, X_4) \quad (29)$$
$$R_{21} \leq I(X_2; Y_1, Y_3|X_1, X_3) \quad (30)$$
$$R_{34} \leq I(X_3; Y_2, Y_4|X_2, X_4) \quad (31)$$
$$R_{43} \leq I(X_4; Y_1, Y_3|X_1, X_3) \quad (32)$$
$$R_{12} + R_{34} \leq I(X_1, X_3; Y_2, Y_4|X_2, X_4) \quad (33)$$
$$R_{21} + R_{43} \leq I(X_2, X_4; Y_1, Y_3|X_1, X_3) \quad (34)$$
$$R_{12} + R_{43} \leq I(X_1, X_4; Y_2, Y_3|X_2, X_3) \quad (35)$$
$$R_{21} + R_{34} \leq I(X_2, X_3; Y_1, Y_4|X_1, X_4) \quad (36)$$

where X_1, X_2, X_3, X_4 follow the fully general input distribution of $p(x_1, x_2, x_3, x_4)$. Although it is hard to compare the single-rate bounds (29) – (32) with our bounds (1) – (4), our sum-rate bounds (5) – (12) may intuitively potentially improve upon the cut-set bounds (33) – (36) due to the presence of auxiliary random variables $Z_i, i \in \{1, 2, 3, 4\}$ and the more constrained input distributions, though this is in general an open problem and the subject of ongoing work.

B. Comparison with Zhang and Berger's [12] outer bound for the two-way channel

Theorem 1 of [12] yields the following outer bound to the capacity region $\mathcal{R}$ of the discrete memoryless two-way channel:

$$\begin{aligned}\mathcal{R}^* \equiv \{(R_1, R_2) :\\ &R_1 \leq \min[H(X_1|Z_1), I(X_1; Y_2|X_2, Z_2)],\\ &R_2 \leq \min[H(X_2|Z_2), I(X_2; Y_1|X_1, Z_1)]\}\end{aligned}$$

where $Z_1, Z_2, X_1, X_2, Y_1, Y_2$ are random variables whose joint distribution is of the form $p(z_1, z_2)p(x_1|z_1)p(x_2|z_2)p(y_1, y_2|x_1, x_2)$.

As the outer bounds of Theorem 1 are based on the techniques of [12] it is natural to expect that they would reduce to [12] for a single two-way channel. Indeed, our outer bound region may be reduced to the above region by setting $X_3, X_4, Y_3, Y_4, M_{34}, M_{43} = \emptyset$. We note that the first term in (1) and (2) is always larger than the third term (and thus the third term dominates the $\min(\cdot, \cdot, \cdot)$, and the third term corresponds to the outer bound of [12]. We note that our bounds $R_{12} \leq H(X_1|Z_1, Z_2, X_2)$ and $R_{21} \leq H(X_2|Z_1, Z_2, X_1)$ are no looser than Zhang and Berger's bounds $R_1 \leq H(X_1|Z_1)$ and $R_2 \leq H(X_2|Z_2)$.

V. Conclusion

In this paper we proposed an outer bound region for the discrete memoryless parallel two-way channel with interference which utilizes auxiliary random variables to constrain the input distribution. Besides the derivation of inner bounds to the capacity region of the parallel two-way channel with interference, a number of remaining outer bound questions are the subject of ongoing work, including: if/when is this bound tighter than the cut-set bound, and whether it may be tightened for specific channel models including deterministic, linear high-SNR deterministic, or Gaussian channels.

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