Diversity-Multiplexing Tradeoff

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Abstract

In this paper, we focus on the diversity-multiplexing tradeoff (DMT) in MIMO channels and introduce the development of DMT in MIMO channels. Zheng and Tse first proposed DMT in MIMO channels and based on their ideas, other researchers extended to many different aspects. We will study DMT in different MIMO fading channels, DMT curves and DMT under different protocols.

I. INTRODUCTION

The diversity-multiplexing tradeoff (DMT) provides a theoretical background for multiple-antenna systems and multiple antennas are very important to improve the performance of wireless systems. Compared with the conventional single-antenna channels, the spectral efficiency is much higher in MIMO channel. DMT establishes the basis tradeoff between reliability and rate via diversity gain and spatial multiplexing gain which are two types of gains in a MIMO channel. Diversity gain is a measure of reliability, and shows how fast the error probability decays with increasing signal-to-noise ratio (SNR) while spatial multiplexing gain is related to the transmission rate of the system, and shows how this rate increases with increasing SNR. In the past, most researches put the attention on designing schemes to get maximal diversity gains or maximal spatial multiplexing gains. However, when we maximize one type of the two gains, we simply cannot maximize the other one.

In 2003, in order to solve this problem, Zheng and Tse proposed a new idea that for a MIMO channel, the diversity gain and spatial multiplexing gain can be obtained simultaneously, but there exists a fundamental tradeoff between how much of each type of gain any coding scheme can extract [1]. Further researches which will be shown in this report are focusing on how we extended the basic idea proposed by Zheng and Tse to different channels: Multiple-access channels, ARQ channels, correlated Rayleigh and Rician MIMO channels, fading relay channels, Half-Duplex ARQ relay channels, ordered MIMO SIC receivers and MIMO relay channels for a broad class of fading distributions. All of these researches are related to the work proposed by Zheng and Tse.

The rest of the paper is outlined as follows: Section II introduces the work of Zheng and Tse where we will introduce the main idea of DMT. In section III, we introduce further researches on DMT in different fading channels which are extended from Zheng and Tse’s original paper. Section IV is conclusions.

II. DIVERSITY-MULTIPLEXING TRADEOFF

In this section, what is DMT will be introduced, along with the codes introduced by Zhang and Tse to achieve the optimal DMT [1].

What we are going to focus on is the high-SNR regime, and we will consider a scheme as a family of codes for different SNR levels. Let’s suppose that in this scheme we have a diversity advantage $d$ and a spatial multiplexing gain $r$. It is known that if the path between individual antenna pairs are i.i.d. Rayleigh faded and also with the assumption that this system has $m$ transmit antennas and $n$ receive antennas, the maximal diversity gain will be $mn$. Considering the same system, the number of degrees of freedom is $\min(m, n)$ and $r$ cannot go over the total number of degrees of freedom. There is an optimal diversity advantage $d^*(r)$ for each $r$ in optimal tradeoff curve. We also have the knowledge that $d^*(r)$ cannot exceed the maximal diversity gain $mn$. Thus, when $d = 0$, $r$ equals to $\min(mn)$, and when $d = mn$, $r$ equals to 0. What are stated above are two extreme situations. The tradeoff curve should lie between these two situations.

When we have a maximal diversity gain (advantage) of $n$, the average error probability can decay as $\frac{1}{SNR^r}$. If $r$ is known, we will get the data rate which is $r\log SNR$. Therefore, we can make the conclusion that the diversity-multiplexing tradeoff is the tradeoff between the data rate and error probability of a system.
A. Channel Model

Now, we consider a wireless system with $m$ transmit antennas and $n$ receive antennas. From transmit antenna $j$ to receive antenna $i$ can we define the fading coefficient $h_{ij}$. Assuming the coefficients are independently complex circular symmetric Gaussian with unit variance, we can write that $H = [h_{ij}] \in \mathbb{C}^{n \times m}$. Also, we assume $H$ is known at the receiver, but is not known at the transmitter, which remains constant within a block of $t$ symbols. Then we have:

$$Y = \sqrt{\frac{\text{SNR}}{m}} HX + W$$

where $X$ has entries $x_{mt}$, means the signals transmitted from antenna $m$ at time $t$; $Y$ has entries $y_{nt}$, means the signals received from antenna $n$ at time $t$. Also, we assume $W$ has i.i.d. entries $w_{nt}$ here. The channel model is shown in Fig. 1:

![Channel Model Diagram]

Now, let a scheme $\{C(\text{SNR})\}$ have the block length $l$ and let $R(\text{SNR})$ be the rate of the code $C(\text{SNR})$. If we can have data rate in the following:

$$R(\text{SNR}) = r \log \text{SNR}$$

Then we can say that this scheme achieves a spatial multiplexing gain of $r$. Zheng and Tse have given us the following definition in their paper.

**Definition 1:** A scheme $\{C(\text{SNR})\}$ is said to achieve spatial multiplexing gain $r$ and diversity gain $d$ if the data rate

$$\lim_{\text{SNR} \to \infty} \frac{R(\text{SNR})}{\log \text{SNR}} = r$$

and the average error probability

$$\lim_{\text{SNR} \to \infty} \frac{\log P_e(\text{SNR})}{\log \text{SNR}} = -d$$
For each $r$, define $d^*(r)$ to be the supremum of the diversity advantage achieved by any scheme. As we can see from the definition above, the SNR can scale the data rate and the error probability, which allows us to see the tradeoff in a more meaningful way.

**B. Optimal Tradeoff Curve**

Let's consider the case when $t \geq m + n - 1$. The following is the main result which Zheng and Tse gave us in their paper.

*Theorem 1:* Assume $t \geq m + n - 1$. The optimal tradeoff curve $d^*(r)$ is given by the piecewise linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, \min\{m, n\}$, where

$$d^*(k) = (m - k)(n - k)$$

Particularly, $d^*_{\text{max}} = mn$ and $r^*_{\text{max}} = \min\{m, n\}$. The function of $d^*(r)$ is shown as follows.

![Diversity-multiplexing tradeoff](image)

As we can see from the figure shown above, there are two extreme conditions where one is the optimal tradeoff curve intersecting the axis $r$ at $\min\{m, n\}$. This means the total number of degrees of freedom is the maximum spatial multiplexing gain $r^*_{\text{max}}$ that we can achieve. However, the diversity gain $d$ is 0 at this point. The other extreme condition is when the optimal tradeoff curve intersects the axis $d$ at $mn$, which is the maximum diversity gain. Also, the spatial multiplexing gain $r$ is 0 at this point.

We can get one optimal tradeoff curve $d^*(r)$ by connecting the two extreme points. From Fig. 2, we can see that both positive diversity gain and spatial multiplexing gain can be achieved at the same time. What is more, the optimal tradeoff curve is a decrease function, which means the spatial multiplexing gain will decrease when we increase diversity gain. Using tradeoff curve can we get a more comprehensive picture of the achievable performance over multiple-antenna channels than only considering the two extreme points. If we increase $r^*_{\text{max}}$ by one, the channel will gain one more degree of freedom. If we increase both $m$ transmitters and $n$ receivers by one, the entire tradeoff curve will be shifted to the right by 1. For any given diversity gain requirement $d$, the spatial multiplexing gain is increased by 1.

**III. FURTHER RESEARCHES ON DMT**

Tse extended the result which is in [1] from the point-to-point context to the many-to-one context in [2]. Compared with [1] where the tradeoff is between two types of gains (diversity gain and spatial multiplexing...
gain), Tse provided a full picture of the tradeoff among three types of gains (diversity gain, spatial multiplexing gain and multiple-access gain).

When several users communicate to a common receiver, the multiple receive antennas also allow the spatial separation of the signals of the different users, which gives us a multiple-access gain. Since it is based on the spatial separation, this method is thus called space-division multiple access (SDMA).

We consider the i.i.d. Rayleigh-fading multiple access channel with $K$ users for the many-to-one case where for each user, it has $m$ transmit antennas. But since we have only one receiver here, it has $n$ receive antennas for this receiver. And it has a multiplexing gain $r_i$ for user $i$. We can know that the data rate $R_i$ equals to $r_i \log \text{SNR}$ from the equation in [1]. Let’s make the diversity gain is $d$ and what we want is to make the minimal error probability decay at the speed no less than $\text{SNR}^{-d}$. In this case, Tse gave us a basic result of optimal tradeoff.

**Theorem 2:** If the block length $l \geq Km + n - 1$

$$\mathcal{R}(d) = \{ (r_1, \ldots, r_K) : \sum_{s \in S} r_s \leq r^*_{S, m, n}(d), \forall S \subseteq \{1, \ldots, K\} \}$$

where $r^*_{S, m, n}(\cdot)$ is the multiplexing-diversity tradeoff curve for a point-to-point channel with $S \cdot m$ transmit and $n$ receive antennas.

Gamal extended the diversity-multiplexing tradeoff proposed by Zheng and Tse in standard delay-limited MIMO channels to the automatic retransmission request (ARQ) channels in [3]. The channel state information (CSI) is not considered in Zheng-Tse system model, which means the transmitter does not have the CSI. While in Gamal’s work, he extended it to ARQ MIMO channel and we have a one-bit success/failure indicator from receiver to the transmitter which is a feedback from receiver to transmitter. The transmitter will transmit the next message if the receiver feeds back a success indicator. Similarly, if the receiver feeds back a failure indicator, the transmitter will transmit the same message again.

In Gamal’s paper, he gave us the result on the diversity-multiplexing tradeoff of MIMO ARQ channel as follows:

**Theorem 3:** The optimal diversity gain of the coherent block-fading MIMO ARQ channel with $M$ transmit, $N$ receive antennas, maximum number of ARQ rounds $L$, under the short-term power constraint, is given as follows:

In the case of long-term static channels

$$d^*_{ls}(r_e, L) = \begin{cases} f(\frac{r_e L}{L}), & 0 \leq r_e < \min\{M, N\} \\ 0, & r_e \geq \min\{M, N\}. \end{cases}$$

In the case of short-term static channels,

$$d^*_{ss}(r_e, L) = \begin{cases} Lf(\frac{r_e L}{L}), & 0 \leq r_e < \min\{M, N\} \\ 0, & r_e \geq \min\{M, N\}. \end{cases}$$

Furthermore, the optimal tradeoff is achieved by codes with finite block length $T$ subject to the conditions

$$T \geq \left\lfloor \frac{M + N - 1}{L} \right\rfloor, \quad \text{for long-term static channels}$$

$$T \geq M + N - 1, \quad \text{for short-term static channels}$$

In fact, the ARQ diversity gain also appears in long-term static channels. The tradeoff curve become flatter when $L$ increases as we can see in Fig. 3, which means that the full diversity point can be approached.

Narasimhan proposed a nonasymptotic framework to analyze the diversity-multiplexing tradeoff of a MIMO system with finite SNR in [4]. In [1], the tradeoff can be obtained when SNR approaches infinity, and the asymptotic characterization is typically valid for both data rates and low error rates. However, in wireless local
area networks (WKANs) system, we have relatively moderate target packet error rates (PERs) around $10^{-2}$ to $10^{-1}$. To solve this problem, Narasimhan proposed a nonasymptotic, finite-SNR analysis of the diversity-multiplexing tradeoff in MIMO systems for realistic propagation conditions and the author computed the diversity gains as functions of the multiplexing gain and SNR for correlated Rayleigh fading and Rician fading. We can use the finite-SNR diversity gain to estimate the additional power to decrease the error probability more precisely.

Also, Zhao extended the result of [1] to a generalized fading channel conditions in [5]. Even though we have derived the optimal tradeoff result under the i.i.d. Rayleigh fading conditions in [1], Rayleigh fading cannot model all the channel conditions accurately since up to then a system is with $n_T$ transmit, $n_R$ receive antennas and a block length $l$. So, Zhao conclude the optimal tradeoff curve in the theorem as follows:

**Theorem 4:** For an $(n_T, n_R, l)$ system with independent and regular fading paths and $l \geq n_T + n_R - 1 + \sum_{i,j} t_{ij}^2$, the optimal tradeoff curve $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, \min\{n_R, n_T\}$, where

$$d^*(k) = \begin{cases} (n_R - k)(n_T - k), & \text{if } k = 1, \ldots, \min\{n_R, n_T\} \\ \sum_{j=1}^{n_R} \sum_{j=1}^{n_T} (1 + t_{ij}^2), & \text{if } k = 0 \end{cases}$$

The method proposed by Zhao also can be used to analyze the diversity-multiplexing tradeoff in multiple-access and broadcast channels.

Stauffer analyzed the diversity-multiplexing tradeoff in a fading relay channel in [6]. The estimates of the diversity and multiplexing gains of the fading relay channels were derived for finite-SNR. They also derived the exact closed form for diversity and multiplexing gains with finite-SNR for time division multiple access (TDMA). For the relay nodes, the role of them in the diversity-multiplexing tradeoff were found as well. What is more, their results show us what is the role of the source-to-destination link to determine finite-SNR diversity-multiplexing tradeoff performance.

In [7], Tabet brought forward a framework for the delay-limited fading ARQ single relay half-duplex channel, where the author studied two kinds of diversity which are space diversity available through the relay terminal and the ARQ diversity obtained by leveraging the retransmission delay to enhance the reliability. And the diversity multiplexing delay tradeoff is shown as follows:

**Theorem 5:** The optimal diversity-multiplexing-delay tradeoff for the ARQ relay channel for the long-term static and short-term static relay channel is
subject to the constraint that $TL \geq 2$ for the long-term static channel and $T \geq 2$ for the short-term static channel, $0 \leq r_e \leq 1$.

We can see from the above statements that even though the optimal tradeoff curves for MIMO channels have been explored, the tradeoff curves for some suboptimal and practical MIMO schemes are still open to be found. In [8], Zhang and Dai proposed a geometrical approach that verified the optimal ordering rule for a V-BLAST SIC receiver will not improve its performance regarding diversity-multiplexing tradeoff in point-to-point channels under general settings.

As in [1], Zheng and Tse have shown for a given family of codes, it was possible that they can be multiplexing and diversity optimal and the optimal codes could be used to achieve operating points on the optimal tradeoff. In [9], Stauffer combine the diversity-multiplexing tradeoff and the diversity-code rate tradeoff into the code rate-diversity-multiplexing tradeoff which is done by constructing the performance bound and showing it achievable. There exists a new tradeoff as the code rate condition is not fulfilled and this is the reason why some MIMO coding schemes for multiplexing gain were not the optimal as well as useful for getting the limits of a MIMO channel.

Next generation wireless communications will consist of relaying systems, which can process the overheard information from nearby nodes and pass it to the destination terminal to improve the system performance. An important constraint in relay channels is the half-duplex constraint since wireless devices have to operate in half-duplex mode. In [10], Leveque expressed the half-duplex relay channel DMT as the solution of a minimization problem and specially come up with the solution to the minimization problem when the source in half-duplex mode. In [10], Leveque expressed the half-duplex relay channel DMT as the solution of a minimization problem and specially come up with the solution to the minimization problem when the source in half-duplex mode. In [10], Leveque expressed the half-duplex relay channel DMT as the solution of a minimization problem and specially come up with the solution to the minimization problem when the source in half-duplex mode. In [10], Leveque expressed the half-duplex relay channel DMT as the solution of a minimization problem and specially come up with the solution to the minimization problem when the source in half-duplex mode. In [10], Leveque expressed the half-duplex relay channel DMT as the solution of a minimization problem and specially come up with the solution to the minimization problem when the source in half-duplex mode.

Let us first define $l_0$ to be the minimum of $n$ and $\lfloor \frac{m+1}{2} \rfloor$, then

a) For $0 \leq r \leq \frac{l_0}{2}$, $d_{H,D,0}(\frac{r}{2}) = n^2 + (m - l)(n - l)$, where $l \in \{0, \ldots, l_0\}$.

b) For $\frac{l_0}{2} \leq r \leq n - \frac{l_0}{2}$, $d_{H,D,0}(\frac{r}{2} + l) = l_0^2 + (n + m - l)(n - l_0 - l)$, where $l \in \{0, \ldots, n - l_0\}$.
c) For \( n - \frac{l_0}{2} \leq r \leq n \), \( d_{HD,0}(n - \frac{l}{2}) = l^2 \), where \( l \in \{0, \ldots, l_0\} \).

The results above are shown in Fig. 4.

![Graphs showing half-duplex DMT curves for different values of m.

In [11], Karmakar put the interest in the three-terminal relay network with the source, relay and destination each equipped with multiple and different number of antennas. By using the result of the joint eigenvalue of two specially correlated random Wishart matrices specify the optimization problem whose solution is its DMT. they got the optimal diversity-multiplexing tradeoff for the three node half-duplex MIMO relay network, where each node operate in the dynamic decode-and-forward protocol (DDF). The optimal diversity order is given by Karmakar in the following:

**Theorem 6:** The optimal diversity order \( d^*(r) \) of the DDF protocol at any multiplexing gain \( r \) is given by

\[
d^*(r) = \min \{ \hat{d}(r) + \varphi(r, t), (d_{m,n}(r) + d_{k,m}(r)) \}
\]

for \( 0 \leq r \leq \min\{m, n\} \), where \( \varphi(\cdot, \cdot) \) is defined in the following:

\[
\varphi(x, y) = \begin{cases}
0 & \text{if } x < y; \\
+\infty & \text{if } x \geq y.
\end{cases}
\]

\( d_{m,n}(r) \) represents the diversity order of a PtP MIMO channel at a multiplexing gain and \( \hat{d}(r) \) is given in the following:

\[
\hat{d}(r) = \min_{1 \leq i \leq 3} \min_{y \in R_i} F(\phi_{r, i}(r-b(1-\frac{r}{y})), \phi_{\beta, i}(r))
\]

Also

\[
F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \triangleq E(\tilde{\alpha}, \tilde{\beta}) + \sum_{i=1}^{t} (k + m - 2i + 1) \gamma_i
\]

\( B_1(y) = [0, \frac{y(n-r)}{r}] \); \( B_2(y) = [0, q] \)

\( B_3(y) = [0, \frac{ry}{(n-r)}]; R_1 = (r, \frac{qr}{n-r}) \)

\( R_2 = (\frac{qr}{(n-r)}, \frac{qr}{(q-r)}]; R_3 = (\frac{qr}{q-r}, t] \)
and the vectors $\phi_\alpha(\cdot)$, $\phi_\beta(\cdot)$ and $\phi_\gamma(\cdot)$ are defined:

$$
\phi_\alpha(a) = [\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_p]^T : \hat{\alpha}_i = (1 - (a - i + 1)^+) , 1 \leq i \leq p \\
\phi_\beta(b) = [\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_q]^T : \hat{\beta}_j = (1 - (b - j + 1)^+) , 1 \leq j \leq q \\
\phi_\gamma(y) = [\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_t]^T : \hat{\gamma}_l = (1 - (y - l + 1)^+) , 1 \leq l \leq t
$$

The closed-form solution for the $(1, k, 1)$ is also given under the DDF protocol which is compared with under SCF protocol as in Fig. 5. The optimal DMT of the DDF protocol on a $(1, k, 1)$ half-duplex relay channel is given as follows:

$$
d^*(r) \triangleq \begin{cases} 
(k + 1)(1 - r) & 0 \leq r \leq \frac{1}{k + 1} \\
1 + k \left(\frac{1 - 2r}{1 - r}\right) & \frac{1}{k + 1} \leq r \leq \frac{1}{2} \\
\frac{1 - r}{1 - r} & \frac{1}{2} \leq r \leq 1
\end{cases}
$$

Fig. 5. DMT comparison of the DDF and SCF protocol on a $(1, 2, 1)$ relay channel.

IV. CONCLUSIONS

In this paper, we have introduced the diversity-multiplexing tradeoff and its further development. In section II, we can see it is better to use diversity-multiplexing tradeoff curve than to use two extreme points. In section III, except for the work of Zheng and Tse, some other papers use diversity-multiplexing tradeoff to analyze the scheme applied to different channels. During the introduction of the development of DMT, we can clearly sense the improvement of DMT and the wide open field for the future researches. Hence, it is both exciting and interesting to investigate DMT on other MIMO channels and under other protocols.

REFERENCES


