Information Theory and Stock Market

Pongsit Twichpongtorn
University of Illinois at Chicago
E-mail: ptwich2@uic.edu

Abstract

This is a short survey paper that talks about the development of important theories in stock market investing by focusing on information theory related works. The very early works on the fundamental problem and some of the very recent works (latest mentioned paper is in 2011) are covered. By using easy to understand language to explain the cause and consequences, this paper is for everyone interested in stock market investing development and has some background knowledge on information theory.

I. INTRODUCTION

Investing on a stock market is one of the ways that people let their money work for them. There are many kinds of stock investing. A popular one is investing on mutual funds. The mutual funds are operated by a professional group of analysts that take the responsibility to invest clients’ money on many categories of stocks. The goal is to make the most profit for their clients. It is very popular because the risk of investing on mutual funds is considerably lower than investing to common stocks directly by nonprofessional investors (as most of people). The interest rate is normally significantly higher than the interest rate that banks offer. They can achieve this by applying many tools to help them decide which stocks they should invest. Those tools have been developed for decades by applying theories to them. In this paper, we will go through the important fundamental theories behind those tools. Later, the information theory related works which play an important role in this field and the latest developments will be discussed.

II. MODERN PORTFOLIO THEORY

In stock markets, a stock price goes up or down everyday. We define the return of a stock as a ratio of the closing price to the opening price of the stock on daily basis. That return is a random variable. Let it be $X_i$. Normally, investors, especially professional investors, do not invest in only one stock at a time because the risk is considered higher than investing on variety types of stock. We can see all the stocks that we are investing on as a stock vector $X$. Then, we have $X = (X_1, X_2, ..., X_m)$ where $m$ is the number stocks. All of the stocks investors buy will be collected as one of their portfolios. If we consider all the money that one has for the investment to be 1 and let $b$ be a portfolio, then the sum of all portfolios would be 1 ($\sum b_i = 1$). We can define how success of the investment by looking at the return(profit or loss) that we get from the investment. This leads us to the definition of the wealth relative [1].

Wealth Relative:

$$S = b^t X = \sum_{i=1}^{m} b_i X_i$$ (1)
The main goal that researchers try to achieve is to maximize this wealth relative because it means that they can maximize the return. A very famous theory about this is “Modern Portfolio Theory” by Markowitz.

A. Modern Portfolio Theory

When investors buy stocks, they always think about how much the profit they will get in return. In general, if the expected return is high, the risk is high also. As a classic phase states that “High risk, high return”. Modern portfolio theory by Markowitz explains how investors should select a portfolio and make the highest possible return from a certain level of risk or get the lowest possible risk for a certain level of return. The expected return and the portfolio return variance are shown as follows [2].

**Expected Return:**

\[
E(R_p) = \sum_i w_i E(R_i)
\]

where \( R_p \) is the return on portfolio, \( R_i \) is the return on asset \( i \) and \( w_i \) is the weighting of component on asset \( i \)

**Portfolio Return Variance:**

\[
\sigma^2_p = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}
\]

where \( \rho_{ij} \) is the correlation coefficient between assets \( i \) and \( j \), \( \rho_{ij} = 1 \) for \( i = j \).

One important thing that the theory introduces is “efficient frontier” which illustrates the relationship between Mean and Variance. On the efficient frontier line are the portfolios that give the highest Mean [2].

![Efficient Frontier](image)

Modern portfolio theory made an impact to the financial industry. Later the creators won a Nobel memorial prize from this theory [4]. It has been used widely and set a foundation for many other theories and models.
B. Capital Asset Pricing Model

Base on modern portfolio theory, a model is developed on the purpose of telling the real price of a stock. This model helps investors to make the decision of buying a stock easier because they know that the price of the stock that they are going to invest is higher or lower than the real price. Certainly, when the price is too low compared to the real price, investors should buy that stock and when the price is too high, investors should sell the stock. By this method, we can ensure that we are able to improve the return. The famous model in question for evaluate the real price of a stock is called “Capital Asset Pricing Model” [3].

Capital Asset Pricing Model:

\[ E(R_i) = R_f + \beta_i(E(R_m) - R_f) \]  

where \( R_f \) is the risk-free interest rate, \( \beta_i = \frac{Cov(R_i, R_m)}{\text{Var}(R_m)} \), \( R_m \) is the return of the market.

Both modern portfolio theory and capital asset pricing model can dramatically improve the way investors invest on a stock market. However, they still have a drawback which is the dependent on mean and variance of portfolios because of two main reasons. First, some important and useful non-Gaussian portfolios distribution have no variances [5]. Second, in reality, investors tend to reinvest more often rather than wait for a very long time to invest. By using the mean of portfolios, it is not so good to tell how the price of a stock should be in a short-term investment [1]. This is where information theory comes to be a part and improve this field.

III. Log-Optimal Portfolio

Instead of using the expected value of a stock as shown in the previous section, information theory defines the growth rate of a stock market portfolio by expected logarithm.

Growth Rate:

\[ W(b, F) = \int \log b' x dF(x) = E(\log b' X) \]  

The portfolio that maximizes the growth rate (\( \max_b W(b, F) \)) is called “log-optimal portfolio”. If the log is base 2, then we can call it “doubling rate” [1]. The idea of log-optimal portfolio is very important and opens the door for information theory to come into this field. It was introduced by Kelly in 1956 [6]. The log-optimal portfolio strategy can give us the maximum return. The proves of the optimality of log-optimal portfolio strategy are provided as follows.

A. Prove of the Optimality of Log-Optimal Portfolio

Kuhn-Tucker characterization and asymptotic optimality of the log-optimal portfolio show that the log-optimal portfolio strategy will do as good as or better than any other casual strategies for a long-term investment on independent and identically distributed stock market. Let \( b^* \) be a portfolio for a stock market \( X \sim F \). This portfolio \( b^* \) is a log-optimal portfolio if and only if it satisfies the following necessary and sufficient conditions [1].

Kuhn-Tucker Characterization of the Log-Optimal Portfolio:
$$E\left( \frac{X_i}{b^i X} \right) = 1 \text{ if } b^i > 0,$$

(6)

$$\leq 1 \text{ if } b^i = 0.$$  

(7)

It is common that investors just pull the money off and reinvest again in some period of time. We define the wealth of the investment after n days as

The Wealth after n days

$$S_n = \Pi_{i=1}^{n} b^i X_i$$  

(8)

where the \(S_n\) is the wealth after n days of using the constant rebalanced portfolio \(b\) [1].

If a portfolio maximizes the wealth after n days \(S^*_n(x^n) = \max_{b \in B} S_n(b, x^n)\), we call it the best constant rebalanced portfolio \((b^*)\) in hindsight. Let \(S^*_n\) be the wealth after n days using the log-optimal strategy \(b^*\) on i.i.d stocks, and let \(S_n\) be the wealth using a causal portfolio strategy \(b_i\) [1]. Then

\[ E \log S^*_n = nW^* \geq E \log S_n \]  

(9)

For doubling rate,

\[ S^*_n = 2^{nW^*} \]  

(10)

Asymptotic Optimality of the Log-Optimal Portfolio:

\[ \lim_{n \to \infty} \sup \frac{1}{n} \log \frac{S_n}{S^*_n} \leq 0 \text{ with probability 1} \]  

(11)

The prove of the optimality of log-optimal portfolio strategy for a short run is available by Bell and Cover, the detail and explanation can be seen in their 1980 paper at [7]. An algorithm for maximizing expected log investment return is discuss in [8]. An adaptive algorithm for log-optimal portfolio is proved to be optimal in the real data of Shanghai Stock Exchanges as shown in [9].

B. Side Information

If we assume that we know some information which tell us more about the direction of the portfolios that we are interested whether they are going to go up or down. Those information should definitely be very useful. We can describe them in term of information theory as “Side Information”. We can evaluate the value of side information by considering how much the information can increase the growth rate. The mutual information is proved to be the upper bound on the increase of the growth rate as shown in [1]. Let side information be \(Y\) and \(\Delta W\) be the increase of the growth rate. Then,

\[ \Delta W = W(X|Y) - W(X) \leq I(X;Y) \]  

(12)

In other words, this shows that the side information decreases the entropy of the stock vectors and increase the growth rate.
C. Stationary Markets

The growth rate for a stationary market (in the sense of time-dependent market process [1]) exists and is equal to

$$W_\infty^* = \lim_{n \to \infty} W^*(X_n | X_1, X_2, ..., X_{n-1})$$  \hspace{1cm} (13)

We have seen the proves which shows the optimality of the log-optimal portfolio strategy and its extensions. However, if we want to know the log-optimal portfolio, we have to know the distribution of the stock vectors which, most of the time, we do not know that in practice. This leads us to the next step. The goal is to find the log-optimal portfolio without knowing the distribution beforehand.

IV. Universal Portfolio

If we know some information about the distribution of the stock vectors, we can use the constant rebalanced portfolio strategy (the investment that using the log-optimal strategy repeatedly everyday). By using the log-optimal portfolio, we can get the best constant rebalanced portfolio which we are able to achieve that only when we completely know the distribution of the stock vectors.

Universal portfolio tells us how well the portfolio that we selected without making any assumption about the distribution of the stock vectors performs compare to the one that uses the best constant rebalanced portfolio strategy and then try to make the portfolio selection without the stock vector distribution to perform as close as the best constant rebalanced portfolio strategy.

A. Finite-Horizon Universal Portfolios

We, first, limit our analysis to a finite number of periods \( (n) \) of investing on a stock market. For a stock market sequence \( x^n = x, ..., x_n, x_i \in \mathbb{R}_+^m \) of length \( n \) with \( m \) assets, let \( S^*_n \) be the wealth achieved by the optimal constantly rebalanced portfolio on \( x^n \), and let \( \hat{S}_n(x^n) \) be the wealth achieved by any causal portfolio strategy \( \hat{b}_i(\cdot) \) on \( x^n \) [1]; Then,

$$\max_{x_1, ..., x_n} \min_{n_1, ..., n_m} \frac{\hat{S}_n(x^n)}{S^*_n(x^n)} = V_n,$$  \hspace{1cm} (14)

where

$$V_n = \left[ \sum_{n_1 + ... + n_m = n} \binom{n}{n_1 + ... + n_m} 2^{-nH(n_1/n, ..., n_m/n)} \right]^{-1}$$  \hspace{1cm} (15)

If there are only two assets \((m = 2)\), then

$$V_n \sim \sqrt{\frac{2}{\pi n}}$$  \hspace{1cm} (16)
B. Horizon-Free Universal Portfolios

In the absence of the limit \( n \), we cannot exactly define the return of the universal portfolio. However, we can determine the upper bound of it which is achieved by the best constant rebalanced portfolio. We can also find the lower bound on two-asset scenario due to the Cover's work in [1]. Let \( \hat{b}_i \) be a \( \mu \)-weighted universal portfolio and \( \mu(b) \) be the initial wealth distribution.

**Universal Portfolio:**

\[
\hat{b}_i = \hat{b}_i(x^{i-1}) = \frac{\int b S_{i-1}(b, x^{i-1}) d\mu(b)}{\int S_{i-1}(b, x^{i-1}) d\mu(b)}
\]

(17)

For \( m = 2 \) and \( d\mu(b) \) the Dirichlet\((\frac{1}{2}, \frac{1}{2})\),

\[
\frac{\hat{S}_n(x^n)}{\tilde{S}_n(x^n)} \geq \frac{1}{2\sqrt{n+1}}
\]

(18)

For all \( n \) and all \( x^n \)

We can also bound the stocks allowing short sale and margin as shown in [10].

C. Applying Universal Portfolio with Real Stock Market Data

From the work of Cover in [13] which applies the universal portfolio with real stock data, it is shown that the universal portfolio typically gives the best return in comparison with the two selected stocks as shown in figure[2] and figure[3].

Fig. 2. Performance of universal portfolio 1 (picture credit [13])
D. Universal Portfolios with side information

We know from the side information section that the side information can increase the growth rate and reduce the entropy. The research about the universal portfolio continues by applying the universal portfolio with the side information. The detail about this can be explored more in [11].

Universal Portfolio with Side Information

\[
\hat{b}_i(y) = \frac{\int bS_{i-1}(b,y)d\mu(b)}{\int S_{i-1}(b,y)d\mu(b)}, \quad i=1, 2, \ldots, y \in \mathcal{Y}
\]  \hspace{1cm} (19)

The comparison between universal portfolio with and without side information can be seen in [12]

E. Universal Portfolios Limitation

The universal portfolios is very robust because it does not require any assumption about the distribution of the stock vectors. However, there are some concerns regarding the usage of the algorithm in practice, for example, the trading cost. Normally, when we buy a stock, we have to pay some transaction fees or some commissions but the universal portfolio algorithm does not take those into account. The constant rebalanced portfolios can maximize the growth rate only in the absence of the trading cost [11].
V. RECENTLY WORKS

There many works that extends the idea of log-optimal portfolio and universal portfolio. We will discuss some of the very recent works in this section.

A. Universal Portfolio with Transaction Costs

This paper is aiming to create a universal portfolio algorithm that takes the transaction cost into account. This is the work of Bean and Singer, in 2010 [14] and 2011 [15]. It shows that by using factor graphs, the insights of Blum and Kalai’s transaction costs algorithm and by fixing the commission on every transaction, they can construct a more sophisticated and better algorithm of universal portfolio under transaction cost.

B. Directed Information in Portfolio Theory

In 2011, the work of Haim H. Permuter, Young-Han Kim and Tsachy Weissman shows that the directed information is an upper bound on the increment in growth rates of optimal portfolio due to causal side information [16]. The directed information from $X^n$ to $Y^n$ can be defined as

$$I(X^n \rightarrow Y^n) := \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1})$$

C. A New Look of Universal Portfolio

In 2011, Mariko Tsurusaki and Jun’ichi Taceuchi proposed a new portfolio strategies for constant rebalanced portfolio and constant Markov portfolio. The new constant rebalance portfolio proposed is defined as

$$\hat{b}_i(x^{i-1}) = \frac{\int bS_{i-1}(x^{i-1}|b)w_{JB}(b)db}{\int S_{i-1}(x^{i-1}|b)w_{JB}(b)db}$$

where $w_{JB}$ is the Jeffreys prior of the Bernoulli model [17].

D. Portfolio Selection Development

In 2011, Bean and Singer developed some portfolio selection algorithms via constrained stochastic gradients and compare the result from those algorithms to the universal portfolio algorithm by using the historical data [18]. The result is shown in figure[4].

E. Universal Portfolio Algorithm in Realistic-Outcome Markets

In 2010, Tavory and Feder showed that the universal portfolio performs much better in some sequences than all possible sequences and suggest the direction of study in realistic algorithms [19].
VI. CONCLUSION

As far as we discussed about this topic we can see that this topic is very interesting and important to the economic and financial industry. The outstanding works by many researchers have pushed information theory and the development of investing on a stock market moving forward. However, the research in this field of information theory is far from settle. It is still wide open and has many potentials. The more we know about this topic the more return we can make from our investment and the better we can improve the economic and financial industry.

REFERENCES


