**BC - channel**

1h, 15min

**Mac Recap**

\[ w_i \rightarrow x_i \rightarrow T \rightarrow (\hat{w}_i, \hat{w}_0) \]

\[ w_o \rightarrow x_o \rightarrow T \rightarrow (\hat{w}_i, \hat{w}_0) \]

**Codebook**

\[(2^{nR_1}, 2^{nR_2}, n)\] for the Mac is on enc(\theta)

\[ X_1 \times W \rightarrow X_2 \]

\[ X_2 \times W_2 \rightarrow X_2 \]

\[ W_1 = \gamma_1, \ldots, \gamma \]

\[ W_2 = \gamma_1, \ldots, \gamma \]

How do we measure the Pe? Average on \( k \) cases

\[ P_e(n) = \frac{1}{2^n (2^{nR_1} + 2^{nR_2})} \sum_{w_i, w_0} P(X_1(w_i) = y_1, w_0) \]

\[ \text{# codewords} \]

\[ P \text{ over all the possible channel realizations} \]
Capacity
\[ 0 \leq R_1 \leq I(X_1; Y_1 | X_2) \]
\[ 0 \leq R_2 \leq I(X_2; Y_1 | X_1) \]
\[ 0 \leq R_1 + R_2 \leq I(X_1, X_2; Y) \]

Some intuition behind \( I(C^*, C^*) \)

Remark

When \( n \to \infty \) the channel transmission increase linearly but the # of messages exponentially!

Why does this make sense?

Achievability

- iid codewords \( P_{X^n} < P_{X^n} \)
- find the average \( R \) over every codebook
- conclude that at least one code can do better than \( R \) average

Converse

- Fano inequality \( R \leq I(W; Y^n) \)
- Data processing inequality \( W^n \rightarrow X^n \)
- chain rule \( X^n \rightarrow X_i \)
- memoryless channel \( X^n \rightarrow X_{i-1} \ldots X_i^n \)
- conditioning reduces entropy
- conditioning reduces entropy
  
  Broadcast Channel
  - In the dual of the MAC-channel
  - Base station for cellular radio
  - Special case of the interference channel

Now for the SC-channel

Channel Model

\[
\begin{array}{c}
W_1 \\
W_2 \\
\text{Enc 1} \\
X \\
\text{Dec 1} \\
Y_1 \\
\text{Dec 2} \\
Y_2 \\
\end{array}
\]

Definition: $D_{1} - BC$

- $X$ input alphabet
- $Y_1$, $Y_2$ output alphabets
- $P_{w_1, w_2 | x}$

$\mathbb{A} ( ( 2^{nR_1}, 2^{nR_2} ), n )$ code as an
1D function
encoding function

\[ x : \{ 1, \ldots, 2^n \} \times \{ 1, \ldots, 2^{nk} \} \rightarrow \{ 1, \ldots, 2^{nK} \} \]

and two decoding functions

\[ \hat{w}_1 : \{ 1, \ldots, 2^n \} \rightarrow \{ 1, \ldots, 2^{nk} \} \]
\[ \hat{w}_2 : \{ 1, \ldots, 2^n \} \rightarrow \{ 1, \ldots, 2^{nk} \} \]

Average probability of error

\[ P_e = P \left( \hat{w}_1(y_1) \neq w_1 \lor \hat{w}_2(y_2) \neq w_0 \right) \]

A rate is achievable when exists a code

\[ (2^{nk}, 2^{nK}, n) \]

such that \( P_e \rightarrow 0 \) as \( n \rightarrow \infty \)

Remark

- transmission means length \( c \) of messages increase exponentially
- I like to start with outer bounds first
I like to start with outer bounds first. What is the one that we consider first?

The bounds for the one to one channel

\[
\begin{align*}
\text{(1)} & \quad R_1 \leq I(Y_1;X) \\
\text{(2)} & \quad R_2 \leq I(Y_2;X)
\end{align*}
\]

We can derive only the bound for \( R_1 \), the bound for \( R_2 \) can be obtained in a symmetrical manner.

\[

\begin{align*}
W R_2 & \leq H(W_2) \\
& = I(W_2;Y_2^n) + H(W_2^n \mid Y_2^n) \\
& \leq I(W_2;Y_2^n) + H \in W \\
& \leq I(W_1;Y_2^n) + I(W_2;Y_1 \mid W_1) + H \in W \\
& = I(W_1, W_2; Y_2^n) + H \in W \\
& \leq I(X^n;Y_2^n) + H \in W \\
& = H(Y_i^n) - H(Y_i^n \mid X^n) + H \in W \\
& = H(Y_i^n) - \sum_i H(Y_i \mid Y_{i-2}^1, X^n) + H \in W \\
& \leq H(Y_i^n) - \sum_i H(Y_i \mid X^n) + H \in W
\end{align*}
\]
We want to bound the sum of the form

\[
|I_2| \leq \max \max_{1 \leq i \leq n} \left| \frac{1}{x} \right|
\]

Now we have

\[
|I_2| \leq \max_{1 \leq i \leq n} \left| \frac{1}{x} \right|
\]

We obtain

\[
|I_2| \leq \max_{1 \leq i \leq n} \left| \frac{1}{x} \right|
\]

and get

\[
\sum_{x} \left| I \right| \leq \max_{1 \leq i \leq n} \left| \frac{1}{x} \right|
\]

(Continue with additional details)
Every $p_x$. Let’s see

\[ J \triangleq \text{extra collaboration} \]

the sum rate is small when the sum rate when the two encoders are collaborating.

We can do this because both messages tend to be decoded correctly to have a small average probability of error. So we have

\[ N (R_1 + R_2) = H(W_1, W_2) \]

\[ \leq I(W_1; W_2, Y_1^*, Y_2^*) + H(W_1, W_2 | Y_1^*, Y_2^*) \]

\[ \leq I(W_1; W_2, Y_1^*, Y_2^*) + H(Y_1^*, Y_2^*) \]

\[ \leq I(X^n; Y_1^*, Y_2^*) + H(Y_1^*, Y_2^*) \]

\[ \leq H(Y_1^n, Y_2^n) - H(Y_1^n, Y_2^n | X^n) \]

\[ \leq H(Y_1^n, Y_2^n) - H(Y_1^n | X^n) - H(Y_2^n | X^n, Y_1^n) \]

\[ = H(Y_1^n, Y_2^n) - \sum H(Y_1^n | X^n, Y_2^{n-1}) \]

\[ \leq H(Y_1^n, Y_2^n) - \sum \left( H(Y_1^n | X^n) + H(Y_2^n | X^n, Y_1^n) \right) \]

\[ \leq H(Y_1^n) + H(Y_2^n | X^n) \]

\[ \leq H(Y_1^n) + H(Y_2^n) \]
\[ I(Y_1, Y_2; X) = I(Y_1; X) + I(Y_2; X | Y_1) - I(Y_1, Y_2; X) \]

\[ \geq \frac{1}{N} \sum_{i=1}^{N} I(Y_{2i}; X | Y_{1i}) \]

\[ \geq \frac{1}{N} \sum_{i=1}^{N} I(Y_{2i}; X | Y_{1i}) \]

\[ \geq \frac{1}{N} \sum_{i=1}^{N} I(Y_{1i}, Y_{2i}; X) \]

Now the three bounds

\[ R_1 \leq I(Y_1; X) \]

\[ R_2 \leq I(Y_2; X) \]

\[ R_1 + R_2 \leq I(Y_1, Y_2; X) \]

are valid bounds for each input distribution

\[ p(x_1, x_2) \]

in principle we have to consider every possible distribution of the inputs and take the union of all the regions described by these inequalities.

This is looking ugly... we need a break!
\[ n \sum_{i} I(y_{1i}, x_i) \leq \sum_{i} I \left( \frac{1}{n} \sum_{i} y_{1i}, x_i \right) \]

\[ = n \sum_{i} I \left( \frac{1}{n} \sum_{i} y_{1i}, \bar{x}_i \right) \]

\[ = n \sum_{i} I \left( \frac{1}{n} \sum_{i} y_{1i}, \bar{x}_i \right) \]

This follows from the convexity of the mutual information and by setting

\[ P_{\bar{x}, \frac{1}{n} \sum_{i} y_{1i}, \bar{y}_n \left( \Omega_1, c \right)} = \left[ \frac{1}{n} \sum_{i} P_{x_i} \left( \Omega_1 \right) \right] \cdot P_{\bar{x}, \frac{1}{n} \sum_{i} y_{1i}, \bar{y}_n \left( \Omega_1, c \right)} \]

for all the appropriate \( \Omega_1 \)

Remember that \( I(X; Y) \) is concave on \( p_x \)

for fixed \( p(y|x) \)

With \( n = 2 \)

\[ I(\bar{x}, \bar{y}) \]

\[ I(x, y) \]

\[ I(x, y) \]

\[ P_{x_1} \]

\[ P_{x_2} \]
This works also for higher dimensions.

Thanks to this trick we can write

\[
\begin{align*}
\bigcup & \quad \{ R_x \leq I(X; Y_1) \quad \} \\
\bigcap & \quad \{ R_x \leq I(X; Y_2) \\
\bigcap & \quad (R_x + R_2 \leq I(X; Y_1, Y_2) \}
\end{align*}
\]

Is this a good outer bound?

It turns out that the sum rate is too low: allowing full cooperation is too much!

But this outer bound is capacity for a special channel, the DETERMINISTIC-BC-IX

Fix

\[
\begin{align*}
Y_1 &= f_1(X) \\
Y_2 &= f_2(X)
\end{align*}
\]

How can we show this?

We use a particular encoding scheme

\[ \text{BINWNA} \]
Consider the following:

1. Generate $n (R_i, R_i')$ codewords $p_{ui}$ and place them into $\mathcal{E}_i$.

$$\begin{bmatrix}
    w_1 & (w_1, v_i) & \cdots & n_k & w_k \\
    1 & \cdots & 2 & & \\
    4 & \cdots & & \\
    2 & & \\
\end{bmatrix}$$

2. Generate $n (R_2, R_2')$ codewords $p_{u2}$ and place them into $\mathcal{E}_2$.

$$\begin{bmatrix}
    w_2 & (w_2, v_2) & \cdots & n_k & w_k \\
    1 & \cdots & 2 & & \\
    4 & \cdots & & \\
    2 & & \\
\end{bmatrix}$$

**Encoding**

Given $w_1, w_2$, try to find a pair $(v_1, v_2)$ such that $(u, (w_1, v_1), v_2, (w_2, v_2)) \in \mathcal{E}(p_{ui})$.
If more than one, pick one
send \( x = f^n (u, v_2) \)

**Decoder 1**
look for \( \hat{w}_1, \hat{w}_2 \) so that
\[
(u, (\hat{w}_1, \hat{w}_2), \hat{y}_1) \in \mathcal{T}_n (u, y_1)
\]
if there is more than a pair pick 1

**Decoder 2**
same thing

**Possible erases**
1. we cannot find \( \hat{w}_1 \) and \( \hat{w}_2 \)
2. Dec 1 chosen wrong \( \hat{w}_1 \)
3. Dec 2 chosen wrong \( \hat{w}_2 \)

**Analysis**
what is the probability of not finding \( w_1 \) and \( w_2 \)?
\[
\mathbb{P}(w_1 \neq \hat{w}_1, w_2 \neq \hat{w}_2) = \mathbb{P}(w_1 \neq \hat{w}_1) \cdot \mathbb{P}(w_2 \neq \hat{w}_2)
\]

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\[ P_{\text{seq}}(w, w) = P \left[ \bigcap_{n \leq \gamma} \left\{ U(w, \gamma), V_{w}^{n} \right\} \right] \]

10 sequences

\[ = \left[ 1 - P \left( U(w, \gamma) \in T_{w}^{n} \right) \right]^{2} \]

\[ \text{take the limit} \]

\[ \lim_{n \to \infty} P \left( U(w, \gamma) \in T_{w}^{n} \right) \]

\[ = \sum P_{u} \cdot P_{v} \]

\[ u, v \in T_{w}^{n} \]

\[ \geq \sum_{u, v \in T_{w}^{n}} -n(1+\epsilon) \left( H(U) + H(V) \right) \]

\[ \geq (1-\epsilon) \cdot \sum_{u, v \in T_{w}^{n}} -n(1+\epsilon) \left( H(U) + H(V) \right) \]

\[ \geq \frac{n(1-\epsilon)}{2} \frac{H(U) + H(V)}{2} \]
\[ \forall (\alpha-\varepsilon)^2 \leq \frac{4}{\hat{n}(R'_i + R'_e)} \]

Let's go back now:

\[ (1 - (\alpha-\varepsilon))^2 \leq \frac{4}{\hat{n}(R'_i + R'_e)(I(U_i; U_e) + 8)^2} \]

\[ \leq \exp \left( - (\alpha-\varepsilon) \cdot \frac{\hat{n}(R'_i + R'_e - I(U_i; U_e) - 8)}{m} \right) \]

Since \((1-x)^m \leq e^{-mx}\)

So we need

\[ R'_i + R'_e \geq I(U_i; U_e) \]

\[ \text{to make this error small} \]

WLOG assume that \(N_1 = N_2 = 1\)

has been transmitted

Probability of error at Dec 1
\[ R_2 = \mathbb{E}_{X_i \sim Z_i} \mathbb{E}_{Y_i \sim Z_i} \text{PC } U_i Y_i \leq I (U_i, Y_i) \]

\[ \leq 2^{n (R_1 + R'_1)} \mathbb{E}_{U_i \sim T_i} \mathbb{E}_{Y_i \sim T_i} \cdot \frac{1}{2} \sum_{i \neq i} \mathcal{P} (U_i, Y_i) \]

\[ \leq 2^{n (R_1 + R'_1) + H(U_i, Y_i) + \varepsilon} \leq 2^{n (R_1 + R'_1 - I (Y_i, U_i) + \varepsilon)} \]

So \( R_1 + R'_1 \leq I (Y_i, U_i) \)

Similarly for \( R_2 + R'_2 \) we get \( R_2 + R'_2 \leq I (Y_2, U_2) \)

Now we have bounds:

\[ R_1 + R'_1 \leq I (Y_1, U_1) \]
\[ R_2 + R'_2 \leq I (Y_2, U_2) \]
\[ R'_1 + R'_2 \geq I (U_1, j U_2) \]

We can rewrite this as:

\[ R_1 \leq I (Y_1, j U_1) \]
\[ R_1 = I(Y_1; U_1) \]
\[ R_2 = I(Y_2; U_2) \]
\[ R_1 + R_2 \leq I(Y_1; U_1) + I(Y_2; U_2) - I(U_1; U_2) \]

When can we do this?

Back to the deterministic channel!

Now pick \( U_1 = Y_1 \) and \( U_2 = Y_2 \).

Since the channel is deterministic, we can always do that, now the achievable region becomes:

\[ R_1 = H(Y_1) \]
\[ R_2 = H(Y_2) \]
\[ R_1 + R_2 \leq H(Y_1) + H(Y_2) - I(Y_1; Y_2) \]
\[ H(Y_1) + H(Y_2) - H(Y_1) + H(Y_2) - H(Y_1, Y_2) \]
The End