1. Problem 7.2. Additive noise channel (Matteo Carminati). Find the channel capacity of the following discrete memoryless channel:

\[
\begin{array}{c}
X \\
\downarrow
\end{array} + \begin{array}{c}
Z \\
\downarrow
\end{array} \begin{array}{c}
Y \\
\end{array}
\]

Where \( PrZ = 0 = PrZ = a = \frac{1}{2} \). The alphabet for \( x \) is \( X = 0, 1 \). Assume that \( Z \) is independent of \( X \). Observe that the channel capacity depends on the value of \( a \).

Solution:

We can identify two different values for the capacity of the channel according to the value of \( a \).

- First of all, we can note that the outputs of the channel are nonoverlapped if \( a \) has a value different from 1 or \(-1\). Thus, in these cases the channel we are analysing is a noisy channel with nonoverlapping outputs. It is known that the capacity of this channel is 1[bit] since:

\[
C = \max I(X;Y) = \max(H(Y) - H(Y|X)) \quad \text{(mutual information definition)}
\]

\[
= \max(H(Y)) \quad \text{(from nonoverlapping outputs)}
\]

\[
= 1 \quad \text{(max of entropy for a binary variable)}
\]

- In the second case, when \( a = 1 \) or \( a = -1 \), the outputs of the channel can overlap: if \( a = 1 \), \( Y \) can be 1 either if \( X = 0 \) or \( X = 1 \), and if \( a = -1 \), \( Y \) can be 0 either if \( X = 0 \) or \( X = 1 \). Let’s compute the channel capacity for \( a = 1 \), the same value and a similar expression can be found for \( a = -1 \).

\[
H(Y) = -p(y = 0) \log_2 p(y = 0) - p(y = 1) \log_2 p(y = 1) - p(y = 2) \log_2 p(y = 2)
\]

\[
= -\frac{2}{4} \log_2 \frac{1}{4} - \frac{1}{2} \log_2 \frac{1}{2} = 1 + \frac{1}{2}
\]

\[
H(Y|X) = -p(x = 0)H(Y|x = 0) - p(x = 1)H(Y|x = 1)
\]

\[
= \frac{1}{2} + \frac{1}{2} = 1
\]

Thus

\[
C = \max I(X;Y) = \max(H(Y) - H(Y|X)) = 1 + \frac{1}{2} - 1 = \frac{1}{2}
\]
2. **Problem 7.7. Cascade of binary symmetric channels (Davide Basilio Bartolini).** Show that a cascade of \(n\) identical independent binary symmetric channels,

\[
X_0 \rightarrow \text{BSC} \rightarrow X_1 \rightarrow \ldots \rightarrow X_{n-1} \rightarrow \text{BSC} \rightarrow X_n,
\]
each with raw error probability \(p\), is equivalent to a single BSC with error probability \(\frac{1}{2} (1 - (1 - 2p)^n)\) and hence that \(\lim_{n \to \infty} I(X_0; X_n) = 0\) if \(p \neq 0, 1\). No encoding or decoding takes place at the intermediate terminals \(X_1, ..., X_{n-1}\). Thus, the capacity of the cascade tends to zero.

**Solution:**

The conditional probability distribution \(p(y|x)\) for each of the BSCs may be expressed by the transition probability matrix \(A\), given by:

\[
A = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}
\]

The transition matrix for the cascade is given by \(A_n = A^n\); it is possible to exploit the singular value decomposition for \(A\) to be able to easily compute \(A^n\):

\[
A = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} = T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1-2p \end{bmatrix} T, \quad \text{where } T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]

\[
A^n = \left[T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & (1-2p)^2 \end{bmatrix} T\right] = \begin{bmatrix} \frac{1}{2} (1 + (1-2p)^n) & \frac{1}{2} (1 -(1-2p)^n) \\ \frac{1}{2} (1 -(1-2p)^n) & \frac{1}{2} (1 + (1-2p)^n) \end{bmatrix}
\]

Hence, the probability of error of the cascade is \(\frac{1}{2} (1 - (1 - 2p)^n)\) and it is equivalent to a single BSC with this probability of error.

Now say that \(E\) is a random variable that indicates whether an error occurs in the cascade channel. \(E\) can assume values in \(\{0, 1\}\) and its distribution is \(Pr\{E = 1\} = \frac{1}{2} (1 - (1 - 2p)^n)\), \(Pr\{E = 0\} = 1 - \frac{1}{2} (1 - (1 - 2p)^n)\) and the capacity of the channel is \(C = I(X_0; X_n) = 1 - H(E)\), when using a uniform input distribution. Note that for \(n \to \infty\), \(Pr\{E = 1\} \to \frac{1}{2}\) and \(Pr\{E = 0\} \to \frac{1}{2}\), so the probability distribution of the error becomes uniform; we can now compute the limit:

\[
\lim_{n \to \infty} I(X_0; X_n) = \lim_{n \to \infty} (1 - H(E)) = 1 - 1 = 0
\]
3. Problem 7.10. Zero-error capacity (Davide Basilio Bartolini). A channel with alphabet 0, 1, 2, 3, 4 has transition probabilities of the form

\[ p(y|x) = \begin{cases} 
1/2 & \text{if } y = x \pm 1 \mod 5 \\
0 & \text{otherwise}
\end{cases} \]

Solution:

(a) When using a uniform input distribution, we have conjunct distribution \( p(x, y) \) shown in Table 1: Conjunct distribution for point (a).

<table>
<thead>
<tr>
<th>x (\backslash) y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
<td>0</td>
<td>1/10</td>
</tr>
<tr>
<td>1</td>
<td>1/10</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
<td>1/10</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
<td>1/10</td>
</tr>
<tr>
<td>4</td>
<td>1/10</td>
<td>0</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
</tr>
</tbody>
</table>

and we can compute the capacity as:

\[
C = I(X; Y) = \sum_x \sum_y p(x, y) \log_2 \frac{p(x, y)}{p(x)p(y)} = \log_2 \frac{1/10}{25} = \log_2 \frac{25}{10} = 1.32 \text{[bits]}
\]

(b) We need to find couples of symbols that yield non-overlapping outputs once passed through the channel. A possible error-free block code with block length 2 is the one with codewords generated according to \((i + 1 \mod 5), 2i + 1 \mod 5\), which yields: 11, 23, 30, 42, 04. The possible outputs when sending these codewords on the channel are: 11 → {00|22}, 23 → {12|14|32|34}, 30 → {21|24|41|44}, 42 → {01|03|31|33} and 04 → {10|13|40|43}; the intersection of any couple of these sets is empty, thus the code is error-free and is able to send five different messages with two uses of the channel, so \(C = \frac{1}{2} \log_2 5 = 1.161\). It can be seen that it is not possible to find six different non-overlapping codewords in a code with block length two, so this is the maximum zero-error capacity achievable with this kind of code. I would say that this is also the maximum zero-error capacity achievable with any block code on this channel.
4. **Problem 7.13. Erasures and errors in a binary channel (Kenneth S. Palacio Baus).** Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be $\epsilon$ and the probability of erasure be $\alpha$, so the channel is follows:

![Channel Diagram]

**Solution:**

(a) **Find the capacity of this channel.**

We have the following channel transition matrix:

<table>
<thead>
<tr>
<th>$X/Y$</th>
<th>0</th>
<th>$\epsilon$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1 - \alpha - \epsilon$</td>
<td>$\alpha$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>1</td>
<td>$\epsilon$</td>
<td>$\alpha$</td>
<td>$1 - \alpha - \epsilon$</td>
</tr>
</tbody>
</table>

Table 2: Channel transition matrix $P(Y|X)$.

To obtain the capacity:

$$C = H(Y) - H(Y|X)$$  \hspace{1cm} (1)

We can compute then $H(Y)$, from the distribution of $Y$:

- $P(Y = 0) = \frac{1}{2}(1 - \alpha - \epsilon) + \frac{1}{2}\epsilon = \frac{1}{2}(1 - \alpha)$
- $P(Y = 1) = \frac{1}{2}(1 - \alpha - \epsilon) + \frac{1}{2}\epsilon = \frac{1}{2}(1 - \alpha)$
- $P(Y = 2) = \frac{1}{2}(\alpha) + \frac{1}{2}(\alpha) = \alpha$
\[ H(Y) = -\frac{1}{2} (1 - \alpha) \log_2 \left[ \frac{1}{2} (1 - \alpha) \right] - \alpha \log_2 (\alpha) - \frac{1}{2} (1 - \alpha) \log_2 \left[ \frac{1}{2} (1 - \alpha) \right] \]  
(2)
\[ = -(1 - \alpha) \log_2 \left( \frac{1}{2} (1 - \alpha) \right) - \alpha \log_2 (\alpha) \]  
(3)

Now we compute \( H(Y|X) \):

\[ H(Y|X) = P[X = 0] H(Y|X = 0) + P[X = 1] H(Y|X = 1) \]  
(4)
\[ = \frac{1}{2} H(1 - \alpha - \epsilon, \epsilon, \alpha) + \frac{1}{2} H(1 - \alpha - \epsilon, \epsilon, \alpha) \]  
(5)
\[ = H(1 - \alpha - \epsilon, \epsilon, \alpha) \]  
(6)

\[ C = H(Y) - H(Y|X) \]  
(7)
\[ = -(1 - \alpha) \log_2 \left( \frac{1}{2} (1 - \alpha) \right) - \alpha \log_2 (\alpha) - H(1 - \alpha - \epsilon, \epsilon, \alpha) \]  
(8)
\[ = -(1 - \alpha)(\log_2 \left( \frac{1}{2} \right) + \log_2 (1 - \alpha)) - \alpha \log_2 (\alpha) - H(1 - \alpha - \epsilon, \epsilon, \alpha) \]  
(9)
\[ = -(1 - \alpha)(\log_2 (1 - \alpha) - 1) - \alpha \log_2 (\alpha) - H(1 - \alpha - \epsilon, \epsilon, \alpha) \]  
(10)
\[ = -(1 - \alpha) \log_2 (1 - \alpha) + (1 - \alpha) - \alpha \log_2 (\alpha) - H(1 - \alpha - \epsilon, \epsilon, \alpha) \]  
(11)
\[ = -(1 - \alpha) \log_2 (1 - \alpha) + (1 - \alpha) - \alpha \log_2 (\alpha) + (1 - \alpha - \epsilon) \log_2 (1 - \alpha - \epsilon) + \epsilon \log_2 (\epsilon) + \alpha \log_2 (\alpha) \]  
(12)
\[ = (1 - \alpha) - (1 - \alpha) \log_2 (1 - \alpha) + (1 - \alpha - \epsilon) \log_2 (1 - \alpha - \epsilon) + \epsilon \log_2 (\epsilon) \]  
(13)

(b) Specialize to the case of the binary symmetric channel \((\alpha = 0)\).

For \(\alpha = 0\) we have:

\[ C = (1 - \alpha) - (1 - \alpha) \log_2 (1 - \alpha) + H(1 - \alpha - \epsilon, \epsilon) \]  
(14)
\[ = 1 - H(1 - \epsilon, \epsilon) \]  
(15)
\[ = 1 - H(\epsilon) \]  
(16)

(c) Specialize to the case of the binary erasure channel \((\epsilon = 0)\).

For \(\epsilon = 0\) we have:

\[ C = (1 - \alpha) - (1 - \alpha) \log_2 (1 - \alpha) + (1 - \alpha - \epsilon) \log_2 (1 - \alpha - \epsilon) + \epsilon \log_2 (\epsilon) \]  
(17)
\[ = (1 - \alpha) - (1 - \alpha) \log_2 (1 - \alpha) + (1 - \alpha) \log_2 (1 - \alpha) + 0 \log_2 (0) \]  
(18)
\[ = 1 - \alpha \]  
(19)
5. **Problem 7.19. Capacity of the carrier pigeon channel (Kenneth S. Palacio Baus).** Consider a commander of an army besieged in a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one letter (8 bits), that pigeons are released once every 5 minutes, and that each pigeon takes exactly 3 minutes to reach its destination.

*Set up an appropriate model for the channel in each of the cases, and indicate how to go about finding the capacity.*

(a) Assuming that all the pigeons reach safely, what is the capacity of this link in bits/hour?

![Figure 1: Problem 7.19a](image)

We have that 12 pigeons will be released within 60 minutes, each one carrying 8 bits, so we have a capacity of $12(8) = 96$ bits/hour.

(b) Now assume that the enemies try to shoot down the pigeons and that they manage to hit a fraction $\alpha$ of them. Since the pigeons are sent at a constant rate, the receiver knows when the pigeons are missing. What is the capacity of this link?

This is an erasure channel, since some of the received bits are lost (a fraction $\alpha$). The capacity for this channel is $1 - \alpha$ but since we are transmitting 8 bits simultaneously the capacity is $8(1 - \alpha)$ bits per pigeon. Since there are 12 birds sent within an hour, the capacity per hour is $12(8)(1 - \alpha)$ bits per hour.
(c) Now assume that the enemy is more cunning and that every time they shoot down a pigeon, they send out a dummy pigeon carrying a random letter (chosen uniformly from all 8-bit letters). What is the capacity of this link in bits/hour?

We know that when the enemy shoot down a pigeon, its arriving chance is reduced by \( \alpha \). Given that each bird is carrying on 8 bits of information, we have \( 2^8 = 256 \) possible entries for each pigeon sent. Thus, when the enemy replaces the pigeon by the dummy one, the chance about recovering the original entry is \( \frac{1}{256} \). Then, the probability of recovering the original information is: \( 1 - \alpha + \alpha \left( \frac{1}{256} \right) \). If the replaced pigeon does not contain the same entry that was shoot down the probability of getting the wrong entry is uniform: \( \frac{\alpha}{256} \). So we have the following channel transition matrix:

<table>
<thead>
<tr>
<th>X/Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>.</th>
<th>.</th>
<th>255</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 - ( \alpha + \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
</tr>
<tr>
<td>1</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>1 - ( \alpha + \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
</tr>
<tr>
<td>2</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>1 - ( \alpha + \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>1 - ( \alpha + \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>1 - ( \alpha + \alpha \left/ 256 \right. )</td>
</tr>
<tr>
<td>255</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>( \alpha \left/ 256 \right. )</td>
<td>1 - ( \alpha + \alpha \left/ 256 \right. )</td>
</tr>
</tbody>
</table>

Table 3: Channel transition matrix problem 7.19.

The channel transition matrix is symmetric, then we can compute the capacity as follows:

\[
C = \max_{p(y)} I(X; Y)
\]

\[
= H(Y) - H(Y|X)
\]

\[
= \log_2 (256) - H(r)
\]

\[
= 8 - H(1 - \alpha + \alpha \left/ 256 \right., \alpha \left/ 256 \right., \alpha \left/ 256 \right., \ldots, \alpha \left/ 256 \right.)
\]

Where \( r \) is a row of the transition matrix. For computing the capacity per hour, \( C \) has to be multiplied by 12, since 12 pigeons are sent within one hour.