Problem 3.1 (Johnson Jonaris Gad Elkarim + Yasaman Keshtkarjahromi)

a) Let X have a probability distribution function f(x)

\[ E(X) = \int_{0}^{\infty} x f(x) \, dx = \int_{0}^{t} x f(x) \, dx + \int_{t}^{\infty} x f(x) \, dx \]

\[ \geq \int_{t}^{\infty} x f(x) \, dx \geq t \int_{t}^{\infty} f(x) \, dx = t \Pr\{X \geq t\} \]

\[ \Pr\{X \geq t\} \leq \frac{E(X)}{t} \]

An example of a random variable which achieves the inequality with equality:

\[ X = \begin{cases} t & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases} \]

b) Y is a R.V. with mean \( \mu \) and variance \( \sigma^2 \)

\[ \Pr\{|Y - \mu| > \epsilon\} = \Pr\{(Y - \mu)^2 > \epsilon^2\} \]

\[ \leq \Pr\{(Y - \mu)^2 \geq \epsilon^2\} \leq \frac{E(Y - \mu)^2}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2} \]

c) \( Z_1, Z_2, \ldots, Z_n \) are an i.i.d. R.V. with mean \( \mu \) and variance \( \sigma^2 \)

\[ S_n = \bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i \]

\[ \Pr\{|\bar{Z}_n - \mu| > \epsilon\} = \Pr\left\{\left(\frac{1}{n} \sum_{i=1}^{n} Z_i - \mu\right)^2 > \epsilon^2\right\} \]

\[ \leq \frac{\text{Var}(\bar{Z}_n)}{\epsilon^2} = \frac{\sigma^2}{n \epsilon^2} \]

Since \( E(\bar{Z}_n) = \mu \) and \( \text{Var}(\bar{Z}_n) = \frac{\sigma^2}{n} \)

Problem 3.3 (Shu Wang)

Cut 1 may result in a piece of size \( \frac{2}{3} \) with probability \( \frac{3}{4} \). Cut 2 may result in a piece of size \( \frac{3}{5} \) with probability \( \frac{1}{4} \). And the size of the cake after \( n \) cuts will only depend on the numbers of Cut 1 and Cut 2.

Suppose we have \( k \) Cut1 and \( n-k \) Cut2, then the final size of the cake is \( X_n = \left(\frac{2}{3}\right)^k \left(\frac{3}{5}\right)^{n-k} \)

So

\[ \lim_{n \to \infty} \frac{1}{n} \log_2 X_n = \lim_{n \to \infty} \left(\frac{k}{n} \log_2 \frac{2}{3} + \left(1 - \frac{k}{n}\right) \log_2 \frac{3}{5}\right) \]

\[ = \frac{3}{4} \log_2 \frac{2}{3} + \frac{1}{4} \log_2 \frac{3}{5} = -0.632 \implies X_n \approx 2^{-0.632n} \]
Problem 3.6 (Davide Basilio Bartolini)
\[
\lim_{n \to \infty} \left( p(X_1, X_2, \ldots, X_n) \right)^{\frac{1}{n}} = \lim_{n \to \infty} 2^{\log_2 \left( p(X_1, X_2, \ldots, X_n) \right)} = 2^{\lim_{n \to \infty} \left( \frac{1}{n} \log_2 p(X_1, X_2, \ldots, X_n) \right)}
\]
\[(A.E.P.) = 2^{-H(X)}
\]

Problem 3.7 (Johnson Jonaris Gad Elkarim)
a) \( N = 100, r = \text{the number of ones in a sequence}, \ \text{prob}(x = 1) = p = 0.005 \)
Number of elements with \( r \leq 3 = m = \sum_{i=1}^{3} \binom{N}{i} = \binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \binom{100}{3} = 166751 \)
Number of needed bits = \( \log_2 m = 18 \)

b) \( P(r) = \binom{N}{r} \times p^r \times (1 - p)^{N-r} \)
Prob of sequences with codeword = \( P_c = P(0) + P(1) + P(2) + P(3) = 0.99833 \)
Prob of sequences with no codeword = \( 1 - P_c = 0.00167 \)

c) Let \( X \) be the random variable that gives the number of ones in the sequence
Since \( X \sim \text{Ber}(p) \) with \( N = 100 \), hence \( \mu = np = 0.5 \) and \( \sigma^2 = np(1-p) = 0.4975 \)
Choose \( \epsilon = 3.5 \)
Hence \( Pr( |X - \mu| \geq 3.5 ) \leq \frac{\sigma^2}{\epsilon^2} = 0.04061 \)

Problem 3.13 (Johnson Jonaris Gad Elkarim)
\( X \sim \{ \) 
\( Pr \{ X = 1 \} = p = 0.6 \)
\( Pr \{ X = 0 \} = q = 1 - p = 0.4 \)
\( \}

a) \( H(x) = -p \log p - q \log q = 0.97095 \)

b)\( n = 25, \ \epsilon = 0.1 \)
The sequences which fall in the set \( A_{\epsilon}^{(n)} \) satisfies:
\[
2^{-n(H(X)+\epsilon)} \leq p(X_1, X_2, \ldots X_{25}) \leq 2^{-n(H(X)-\epsilon)}
\]
\[
8.7155 \times 10^{-9} \leq p(X_1, X_2, \ldots X_{25}) \leq 2.788 \times 10^{-7}
\]
We need to construct a table and compute the following for \( 0 \leq K \leq 25 \):
Number of elements in each sequence containing \( K \) ones = \( \binom{25}{K} \)
Prob of a string with \( K \) ones = \( P^K \times (1-p)^{25-K} \)
\( Pr(K) = \binom{25}{K} p^K \times (1-p)^{25-K} \)
I created a Matlab program to compute these values and found that the \( A_{\epsilon}^{(n)} \) range is when \( 11 \leq K \leq 19 \) which satisfies the condition. Hence we can deduct that :
\[
Pr(A_{\epsilon}^{(n)}) = \sum_{i=11}^{19} \binom{25}{i} Pr(i) = 0.9362
\]
\[
\text{Numb}(A_{\epsilon}^{(n)}) = \sum_{i=11}^{19} \binom{25}{i} = 2636510
\]
c) To compute the smallest set with \( Pr(B_{\epsilon}^{(n)}) = 0.9 \) I have also used the Matlab to add \( Pr(K) \) starting from 25 down;
I found that we need all the strings with \( 13 \leq K \leq 25 \) and only 3680673 elements from the set with 12 ones sequence.
The total number of elements will be $= \sum_{i=13}^{25} \binom{25}{i} + 3680673 = 2045789$

d) Number of intersected elements $= \sum_{i=13}^{19} \binom{25}{i} + 3680673 = 20389483$

$\Pr(\text{intersection}) = \sum_{i=13}^{19} \binom{25}{i} \times Pr(i) + 3680673 \times Pr(i=12) = 0.8706$