1. **Problem 10.14.** *Rate distortion for two independent sources.* Can one compress two independent sources simultaneously better than by compressing the sources individually? The following problem addresses this question. Let \( \{X_i\} \) be i.i.d. \( \sim p(x) \) with distortion \( d(x, \hat{x}) \) and rate distortion function \( R_X(D) \). Similarly, let \( \{Y_i\} \) be i.i.d. \( \sim p(y) \) with distortion \( d(y, \hat{y}) \) and rate distortion function \( R_Y(D) \). Suppose we now wish to describe the process \( \{(X_i, Y_i)\} \) subject to distortions \( Ed(X, \hat{X}) \leq D_1 \) and \( Ed(Y, \hat{Y}) \leq D_2 \). Thus, a rate \( R_{X,Y}(D_1, D_2) \) is sufficient, where

\[
R_{X,Y}(D_1, D_2) = \min_{p(\hat{x}, \hat{y}|x, y): Ed(X, \hat{X}) \leq D_1, Ed(Y, \hat{Y}) \leq D_2} I(X, Y; \hat{X}, \hat{Y})
\]

Now suppose that the \( \{X_i\} \) process and the \( \{Y_i\} \) process are independent of each other.

(a) **Show that:**

\[
R_{X,Y}(D_1, D_2) \geq R_X(D_1) + R_Y(D_2)
\]

**Solution:**

Given:

\[
R_{X,Y}(D_1, D_2) = \min_{p(\hat{x}, \hat{y}|x, y): Ed(X, \hat{X}) \leq D_1, Ed(Y, \hat{Y}) \leq D_2} I(X, Y; \hat{X}, \hat{Y}) \tag{1}
\]

Considering that \( \{X_i\} \) and the \( \{Y_i\} \) are independent of each other we have:

\[
I(X, Y; \hat{X}, \hat{Y}) = H(X, Y) - H(X, Y|\hat{X}, \hat{Y})
\]

\[
= H(X) + H(Y) - H(X|\hat{X}, \hat{Y}) - H(Y|X, \hat{X}, \hat{Y}) \tag{2}
\]

\[
= H(X) + H(Y) - H(X|\hat{X}) - H(Y|\hat{Y}) \tag{3}
\]

\[
\geq H(X) - H(X|\hat{X}) + H(Y) - H(Y|\hat{Y}) \tag{4}
\]

\[
\geq I(X; \hat{X}) + I(Y; \hat{Y}) \tag{5}
\]

Now we obtain:

\[
R_{X,Y}(D_1, D_2) = \min_{p(\hat{x}, \hat{y}|x, y): Ed(X, \hat{X}) \leq D_1, Ed(Y, \hat{Y}) \leq D_2} I(X, Y; \hat{X}, \hat{Y}) \tag{6}
\]

\[
\geq \min_{p(\hat{x}, \hat{y}|x, y): Ed(X, \hat{X}) \leq D_1, Ed(Y, \hat{Y}) \leq D_2} \left[ I(X; \hat{X}) + I(Y; \hat{Y}) \right] \tag{7}
\]

\[
\geq \min_{p(\hat{x}|x): Ed(X, \hat{X}) \leq D_1} I(X; \hat{X}) + \min_{p(\hat{y}|y): Ed(Y, \hat{Y}) \leq D_2} I(Y; \hat{Y}) \tag{8}
\]

\[
\geq R_X(D_1) + R_Y(D_2) \tag{9}
\]

\[
R_{X,Y}(D_1, D_2) \geq R_X(D_1) + R_Y(D_2) \tag{10}
\]
(b) **Does equality hold?**

From part (a), we have that: 

\[ R_{X,Y}(D_1, D_2) \geq R_X(D_1) + R_Y(D_2). \]

Because of the independence assumed between processes \( \{X_i\} \) and \( \{Y_i\} \) we have that:

\[
p(x, y, \hat{x}, \hat{y}) = p(\hat{x}, \hat{y} | x, y)p(x, y) \\
= p(\hat{x} | x)p(\hat{y} | y)p(x)p(y) \\
= p(x, \hat{x})p(y, \hat{y})
\]

Then, for distributions \( p(x, \hat{x}) \) and \( p(y, \hat{y}) \) achieving rate distortions \( R_X(D_1) \) and \( R_Y(D_2) \) we have that the mutual information of the product of the two distributions such that \( p(x, y, \hat{x}, \hat{y}) = p(x, \hat{x})p(y, \hat{y}) \) gives:

\[
R_{X,Y}(D_1, D_2) = \min_{p(\hat{x}, \hat{y} | x, y): \text{Ed}(X, \hat{X}) \leq D_1, \text{Ed}(Y, \hat{Y}) \leq D_2} I(X, Y; \hat{X}, \hat{Y}) \\
= R_X(D_1) + R_Y(D_2)
\]

Hence, the equality holds.

*Now answer the question: Can one compress two independent sources simultaneously better than by compressing the sources individually?*

From the previous result we can see that encoding two independent sources together results the same as encoding each of them independently.
2. Problem 15.1. Cooperative capacity of a multiple-access channel

(a) Suppose that \( X_1 \) and \( X_2 \) have access to both indices \( W_1 \in \{1, 2^n R\}, W_2 \in \{1, 2^n R_2\} \). Thus, the codewords \( X_1(W_1, W_2), X_2(W_1, W_2) \) depend on both indices. Find the capacity region.

Solution:

Since we know that \( X_1 \) and \( X_2 \) have access to both indices \( W_1 \in \{1, 2^n R\}, W_2 \in \{1, 2^n R_2\} \), and that the codewords depend on both indices, \( X_1(W_1, W_2), X_2(W_1, W_2) \), the pair \((X_1, X_2)\) can be seen as a single codeword \( X \) we have that this is equivalent to have a single user channel with alphabet given by \( X_1 \times X_2 \) and indice \( W_1 \times W_2 \). Then, we have a combined rate for both senders as the only bound for the achievable region given by:

\[
R_1 + R_2 \leq C = \max_{p(x_1, x_2)} I(X_1, X_2; Y) \quad (16)
\]

We can achieve this setting \( X_2 = 0 \) so we have a rate pair \((C, 0)\) and also by setting \( X_1 = 0 \) achieving a rate pair \((0, C)\).

(b) Evaluate this region for the binary erasure multiple access channel \( Y = X_1 + X_2, X_i \in \{0, 1\} \). Compare to the noncooperative region.

The operation of this channel is shown in table 1. Then, the alphabet for \( Y \) is \( Y \in \{0, 1, 2\} \).

\[
\begin{array}{ccc}
X_1 : X_2 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 2 \\
\end{array}
\]

Table 1: \( Y = X_1 + X_2 \)

To evaluate this region for the binary erasure multiple access channel \( Y = X_1 + X_2 \) for the
cooperative capacity region we have:

\[ R_1 + R_2 \leq C = \max_{p(x_1, x_2)} I(X_1, X_2; Y) \]  
\[ = H(Y) - H(Y|X_1, X_2) \]  
\[ = H(Y) \]  
\[ \leq \log_2 |Y| \]  
\[ R_1 + R_2 \leq \log_2 (3) \]  

To achieve this capacity we need to set the distribution of the possible inputs to be \( \text{Uniform}(\frac{1}{3}) \) for example by setting: \( p(0,0) = p(1,1) = \frac{1}{3} \) and \( p(1,0) + p(0,1) = \frac{1}{3} \).

When the senders work in non-cooperative mode, we have the region capacity of the binary erasure multiple-access channel. The capacity region is shown in the following figure:

![Figure 1: Capacity Region for problem 15.1b](image.png)
3. **Problem 15.2. Capacity of multiple-access channels.** Find the capacity region for each of the following multiple-access channels:

(a) **Additive modulo 2 multiple-access channel.**  \( X_1 \in \{0, 1\}, \ X_2 \in \{0, 1\}, \ Y = X_1 \oplus X_2. \)

The following table shows the operation \( Y = X_1 \oplus X_2. \)

<table>
<thead>
<tr>
<th>( X_1 ) : ( X_2 )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: \( Y = X_1 \oplus X_2 \)

Here we see that by setting one of the inputs to zero, it is possible achieve a rate of 1 bit per transmission for the other sender, for example setting \( X_2 = 0, \) the output is given by \( Y = X_1. \) Doing the same for \( X_1 = 0, \) gives \( Y = X_2, \) then we have a rate of 1 bit. We can see from the table 2 that the alphabet of the output is the same as the alphabet of the input, therefore to obtain the verify the capacity:

\[
C = \max_{\text{dist}} I(X_1, X_2; Y) \leq H(Y) = \log_2 |Y| = 1
\]

Then, the combined rates \( R_1 + R_2 \) cannot be more than 1 bit, so we obtain a triangular capacity region as shown in figure 2, defined by \( R_1 + R_2 \leq 1. \)

![figure 2: Capacity region for problem 15.2](image)

(b) **Multiplicative multiple-access channel.**  \( X_1 \in \{-1, 1\}, \ X_2 \in \{-1, 1\}, \ Y = X_1 \cdot X_2. \)

The operation of this channel is shown in the following table:

<table>
<thead>
<tr>
<th>( X_1 ) : ( X_2 )</th>
<th>-1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>+1</td>
</tr>
</tbody>
</table>

Table 3: \( Y = X_1 \cdot X_2 \)
We can see that the result is exactly the same as part (a) as well as the capacity region \((R_1 + R_2 \leq 1).\) Notice that the notation has changed, however, we still have a binary notation +1 and −1.
4. **Problem 15.6.** Unusual multiple-access channel. Consider the following multiple-access channel: \( X_1 = X_2 = Y = \{0, 1\} \). If \((X_1, X_2) = (0, 0)\), then \( Y = 0 \). If \((X_1, X_2) = (0, 1)\), then \( Y = 1 \). If \((X_1, X_2) = (1, 0)\), then \( Y = 1 \). If \((X_1, X_2) = (1, 1)\), then \( Y = 0 \) with probability \( \frac{1}{2} \) and \( Y = 1 \) with probability \( \frac{1}{2} \).

The following table illustrate the operation of this channel:

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0, (Pr=1/2) —— 1, (Pr=1/2)</td>
</tr>
</tbody>
</table>

Table 4: Unusual multiple-access channel

(a) **Show that the rate pairs** \((1, 0)\) **and** \((0, 1)\) **are achievable.**

**Solution:**

To achieve the rate pairs \((1, 0)\) and \((0, 1)\) we can set one of the inputs to zero so the other sender will have a rate of 1 bit per transmission. For example setting \( X_1 = 0 \) will yield into \( Y = X_2 \) then, the rate pair \((0,1)\) is achievable. Thus, we can also do the counter case and set \( X_2 = 0 \) to obtain \( Y = X_1 \) achieving the rate pair \((1,0)\). This can be achieved by time sharing, like it shown in the Binary Multiplier channel example of the textbook.

To prove this, we know the capacity region bound for \( R_1 \) is given by:

\[
R_1 \leq I(X_1; Y|X_2)
\]

For example for fixed \( X_2 = 0 \) we have:

\[
R_1 \leq I(X_1; Y|X_2 = 0) \leq H(Y|X_2 = 0) - H(Y|X_1, X_2 = 0) \leq H(Y|X_2 = 0) \leq H(1/2) \leq 1
\]

The result for setting \( X_1 = 0 \) will follow from symmetry.

(b) **Show that for any non-degenerate distribution** \( p(x_1)p(x_2) \), **we have** \( I(X_1, X_2; Y) < 1 \).

A non-degenerate distribution is obtained by \( p(x_1) \neq \{0, 1\} \) and \( p(x_2) \neq \{0, 1\} \). So, we can set an arbitrary distribution like \( Pr(X_1 = 1) = p \) and \( Pr(X_2 = 1) = q \) then know that the region capacity for the combined rate is given by:
\[ R_1 + R_2 \leq I(X_1, X_2; Y) \leq H(Y) - H(Y|X_1, X_2) \] (25)

We have the distribution of \( Y \) is given by:
\[
Y = \begin{cases} 
0, & \text{with probability } (1-p)(1-q) + \frac{1}{2}pq \\
1 & \text{with probability } (1-p)q + (1-q)p + \frac{1}{2}pq
\end{cases}
\]

Then, we can compute the combined rate:
\[
R_1 + R_2 \leq H(Y) - H(Y|X_1, X_2) \leq H \left( (1-p)(1-q) + \frac{1}{2}pq \right) - pqH \left( \frac{1}{2} \right) \leq H \left( (1-p)(1-q) + \frac{1}{2}pq \right) - pq
\] (27) (28) (29)

Here, as \( p, q \in (0, 1) \), the product \( pq > 0 \) and since, the entropy of binary random variable is bounded by 1, we obtain the result:
\[ R_1 + R_2 < 1 \] (31)

(c) Argue that there are points in the capacity region of this multiple-access channel that can only be achieved by timesharing; that is, there exist achievable rate pairs \((R_1, R_2)\) that lie in the capacity region for the channel but not in the region defined by:
\[
R_1 \leq I(X_1; Y|X_2), \quad R_2 \leq I(X_2; Y|X_1), \quad R_1 + R_2 \leq I(X_1, X_2; Y)
\]
for any product distribution \( p(x_1)p(x_2) \). Hence the operation of convexification strictly enlarges the capacity region. This channel was introduced independently by Csiszár and Körner [149] and Bierbaum and Wallmeier [59].

From part (b) we know that for a non-degenerate distribution the combined rate is: \( R_1 + R_2 < 1 \). We also know that the rates \( R_1 \) and \( R_2 \) are bounded by 1: \( R_1 \leq 1 \) and \( R_2 \leq 1 \). For degenerate distributions we have that \( I(X_1; Y|X_2) = 0 \) and \( I(X_2; Y|X_1) = 0 \) yielding rates \( R_1 = 0 \) or \( R_2 = 0 \). The achievable pairs \((R_1, R_2)\) should lie in the region \( R_1 + R_2 < 1 \) (Convex hull).

Hence, the points in the capacity region that can be achieved only by time sharing are those lying on the line \( R_1 + R_2 = 1 \) which defines the triangular capacity region for the rate pairs \((1,0)\) and \((0,1)\), as in part (a).