**Estimation: chapter 4**

**Linear Models**

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**MVUE and linear models**

The minimum variance unbiased estimator (MVUE) is difficult to find (if it exists) in general. One exception to this rule is when the parameters to be estimated are related to the observations in a linear fashion, i.e. we have a linear model.

\[ \mathbf{x} = \mathbf{H}\theta + \mathbf{w} : \]

- \( \mathbf{x} : N \times 1 \) observation vector
- \( \mathbf{H} : N \times p \) known observation matrix with \( N > p \) and rank \( p \)
- \( \theta : p \times 1 \) vector of unknown parameters to be estimated
- \( \mathbf{w} : N \times 1 \) noise vector, \( \sim \mathcal{N}(0, \sigma^2 \mathbf{I}) \)

One way of finding the MVUE is using the CRLB, where we know that \( \hat{\theta} = g(\mathbf{x}) \) is the MVUE if

\[ \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} = \mathbf{I}(\theta) (g(\mathbf{x}) - \theta), \]

Furthermore the covariance of the estimator will be \( \mathbf{I}^{-1}(\theta) \).
Examples

Consider $x[n] = A + Bn + w[n]$, for $n = 0, 1, \cdots , N-1$ where $w[n]$ is WGN and $A, B$ are to be estimated. Formulate the linear model and determine the MVUE and its statistics when estimating $A$ and $B$. What does the rank condition on $H$ mean?

Consider $x[n] = A + r^n + w[n]$ for $n = 0, 1, \cdots , N-1$ where $w[n]$ is WGN, $r$ is known, and $A$ is to be estimated. Find the MVUE and its statistics when estimating $A$.

Examples

Consider the following polynomial curve fitting problem, where we wish to find $\theta_1, \theta_2, \cdots , \theta_p$ so as to best fit the experimental data points $(t_n, x(t_n))$ for $n = 0, 1, \cdots , N-1$ by the polynomial curve

$$x(t_n) = \theta_1 + \theta_2 t_n + \theta_3 t_n^2 + \cdots + \theta_p t_n^{p-1} + w(t_n)$$

for $w(t_n)$ are WGN samples. Find the MVUE and its statistics when estimating $\theta_1, \theta_2, \cdots , \theta_p$. 

Examples

Consider a signal of the form

\[ x[n] = \sum_{k=1}^{M} a_k \cos \left( \frac{2\pi kn}{N} \right) + \sum_{k=1}^{M} b_k \sin \left( \frac{2\pi kn}{N} \right) + w[n], \quad n = 0, 1, \ldots, N - 1 \]

whose Fourier coefficients \( a_1, \ldots a_M, b_1, \ldots, b_M \) we wish to estimate for a known fundamental frequency \( f_1 = 1/N \) and its harmonics \( f_k = k/N \). Find the MVUE and its statistics when estimating \( a_1, \ldots a_M, b_1, \ldots, b_M \).

MVUE for the linear model

If the data follows the linear model specified above (notice the condition on the rank of the matrix \( H \)) then the MVUE is given by

\[ \hat{\theta} = (H^T H)^{-1} H^T x \]

and the covariance matrix of \( \hat{\theta} \) is

\[ C_{\hat{\theta}} = \sigma^2 (H^T H)^{-1} \cdot \]

In fact, we have a full statistical description of the estimator \( \hat{\theta} \) as we know its pdf, which is

\[ \hat{\theta} \sim \mathcal{N} \left( \theta, \sigma^2 (H^T H)^{-1} \right) \]
Examples

Consider \( x[n] = A + Bn + w[n] \), for \( n = 0, 1, \cdots, N - 1 \) where \( w[n] \) is WGN and \( A, B \) are to be estimated. Formulate the linear model and determine the MVUE and its statistics when estimating \( A \) and \( B \). What does the rank condition on \( H \) mean?

Consider \( x[n] = A + r^n + w[n] \) for \( n = 0, 1, \cdots, N - 1 \) where \( w[n] \) is WGN, \( r \) is known, and \( A \) is to be estimated. Find the MVUE and its statistics when estimating \( A \).

Examples

Consider the following polynomial curve fitting problem, where we wish to find \( \theta_1, \theta_2, \cdots, \theta_p \) so as to best fit the experimental data points \( (t_n, x(t_n)) \) for \( n = 0, 1, \cdots, N - 1 \) by the polynomial curve

\[
x(t_n) = \theta_1 + \theta_2 t_n^1 + \theta_3 t_n^2 + \cdots + \theta_p t_n^{p-1} + w(t_n)
\]

for \( w(t_n) \) are WGN samples. Find the MVUE and its statistics when estimating \( \theta_1, \theta_2, \cdots, \theta_p \).
Examples

Consider a signal of the form

\[ x[n] = \sum_{k=1}^{M} a_k \cos \left( \frac{2\pi kn}{N} \right) + \sum_{k=1}^{M} b_k \sin \left( \frac{2\pi kn}{N} \right) + w[n], \quad n = 0, 1, \ldots, N - 1 \]

whose Fourier coefficients \( a_1, \ldots, a_M, b_1, \ldots, b_M \) we wish to estimate for a known fundamental frequency \( f_1 = 1/N \) and its harmonics \( f_k = k/N \). Find the MVUE and its statistics when estimating \( a_1, \ldots, a_M, b_1, \ldots, b_M \).

General linear model

\[ x = H\theta + w : \]
- \( x \): \( N \times 1 \) observation vector
- \( H \): \( N \times p \) known observation matrix with \( N > p \) and rank \( p \)
- \( \theta \): \( p \times 1 \) vector of unknown parameters to be estimated
- \( w \): \( N \times 1 \) noise vector, \( \sim \mathcal{N}(0, C_w) \)

Re-do the derivation

**Key idea:** *whiten* the observations, then apply previous results
MVUE for general linear model

Under the general linear model, by “whitening” ($x' = Dx, H' = DH, w' = Dw$ where $C^{-1} = D^T D$ for $D$ invertible) we see that the MVUE is given by

$$\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} x$$

and the covariance matrix of $\hat{\theta}$ is

$$C_{\hat{\theta}} = \sigma^2 (H^T C^{-1} H)^{-1}.$$

What if $x = H\theta + s + w$, where $s$ is a known deterministic signal?

Examples

Consider $x[n] = A + w[n]$, for $n = 0, 1, \ldots, N - 1$ where $w[n]$ is colored Gaussian noise with covariance matrix $C$ and $A$ is to be estimated. Formulate the general linear model and determine the MVUE and its statistics when estimating $A$. 