1. (30 points) You are a lawyer. In a community near Chicago, there is the suspicion that the ground-water has been polluted by a large chemical company. Two hypotheses exist: under hypothesis $H_1$ the ground-water is polluted and will cause cancer, and under hypothesis $H_0$ the ground-water is fine and drinkable. Assume that the results of tests on the ground-water obtained from a series of pilot wells in the area are summarized by the value of a random variable $X$. If the ground-water is polluted the random variable $X$ has the following pdf $p(x; H_1) = xe^{-x}u(x)$ where $u(x)$ is the unit step function (0 for $x < 0$ and 1 for $x \geq 0$). In the case that the ground-water is fine the random variable $X$ obeys the following pdf: $p(x; H_0) = e^{-x}u(x)$. You must decide, based on the value of $X$ that is obtained from the well tests, whether or not to initiate legal action against the chemical company.

(a) Design a hypothesis testing rule (find decision regions) such that, $P_D$, the probability that you decide to sue given that the ground-water is polluted is present is maximized subject to the constraint that the probability of suing given the ground-water is fine is 0.1.

(b) Find the resulting value of $P_D$.

(c) Assuming the probability that the water is polluted is 0.1, find the test that will yield the minimum probability of error.

(d) What is the resulting probability of error for part (c)?
2. (20 points) When a RAM chip is selected from a particular production batch there is a probability $\alpha$ that the RAM chip is defective. Thus, $k$, the number of defective RAM chips out of $N$ independently selected ones is a binomial random variable:

$$Pr[k = K] = \frac{N! \alpha^K (1 - \alpha)^{N-K}}{K!(N-K)!}$$

Suppose that we receive a shipment of $N$ RAM chips which either all came from batch 0 (probability of defect $\alpha = \alpha_0$) or all came from batch 1 (probability of defect $\alpha = \alpha_1$), with a priori probabilities $P_0$ and $P_1$, respectively. Assume $0 < \alpha_0 < \alpha_1 < 1$. We wish to determine which production batch our RAM chips came from by testing how many of the $N$ units are defective. Let $x$ denote the number of defective RAM chips out of the total of $N$ RAM chips.

(a) Find the minimum probability of error rule for deciding whether the received RAM chips were from batch 0 or from batch 1.

(b) Suppose $N = 2$ and plot the ROC. (HINT: since $x$ takes on discrete values, the ROC consists of a number of points).
3. (30 points) Suppose we want to determine which of two channels is being used for communications. We observe the output of the channel $x(n)$ for $n = 0, 1, \ldots, N - 1$. We know that

If channel 1 is used: $x(n) = a_1 s_1(n) + w_1(n)$
If channel 2 is used: $x(n) = a_2 s_2(n) + w_2(n)$

(a) Suppose $a_1, a_2, s_1(n), s_2(n)$ are all known and that $w_1 \sim N(0, \sigma^2 I), w_2 \sim N(0, \sigma^2 I)$. Describe the detector that minimizes the probability of error.

(b) Suppose $a_1, a_2, s_1(n), s_2(n)$ are all known and that $w_1 \sim N(0, C_{W_1}), w_2 \sim N(0, C_{W_2})$. Describe the detector that minimizes the probability of error (you do not need to simplify).

(c) Suppose $a_1, a_2$ are unknown, $s_1(n), s_2(n)$ are known and $w_1 \sim N(0, C_{W_1}), w_2 \sim N(0, C_{W_2})$. Determine the generalized likelihood ratio test (again, you do not need to simplify).
4. (20 points) TRUE or FALSE:

(a) The Neyman-Pearson test yields the best probability of detection for a given probability of false alarm regardless of the noise statistics.

(b) When $x \sim \mathcal{N}(0, \sigma_i^2)$ under $\mathcal{H}_i$, $(i = 0, 1)$, the decision region depend only the ratio of $\sigma_0^2/\sigma_1^2$.

(c) The Bayesian risk formulation captures the maximum a posteriori, maximum likelihood and Neyman-Pearson detectors.

(d) When detecting which of multiple known signals was sent in white Gaussian noise the test statistic always boils down to a minimum distance test.

(e) The performance of a matched filter in colored noise does not depend on the signal shape.

(f) When detecting a random Gaussian signal in Gaussian noise, the test statistic is linear in the received data.

(g) Detecting a sinusoid of unknown amplitude, phase and frequency in white Gaussian noise may be cast in the linear model.

(h) The generalized likelihood ratio test outperforms the Rao and Wald tests when detecting a signal with unknown parameters when you have a finite number of samples.