NAME:

- This exam has 4 questions.
- You will be given the full class time: 75 minutes minutes.
- You may use the 2 course textbooks.
- No calculators are permitted.
- No talking, passing notes, copying (and all other forms of cheating) is permitted.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
- If something has been proven in class or in the book feel free to cite and use the result without a re-derivation.
1. Suppose we observe a random $N$-dimensional vector $\mathbf{x}$ (column vector) whose components are independent, identically distributed Gaussian random variables $\mathcal{N}(\mu, \sigma^2)$.

(a) Suppose $\mu$ is unknown but $\sigma^2$ is known (we know the variance but not the mean).

- Does an efficient estimator exist?
- Find the maximum likelihood (ML) estimate of $\mu$.
- Evaluate the bias and the variance of the ML estimate.

(b) Suppose $\mu$ is known but $\sigma^2$ is not (we know the mean but not the variance).

- Does an efficient estimator exist?
- Find the maximum likelihood (ML) estimate of $\sigma^2$.
- Evaluate the bias and the variance of the ML estimate.

(b) Suppose both $\mu$ and $\sigma^2$ are unknown.

- Find the maximum likelihood (ML) estimate of the vector of unknown parameters $[\mu \ \sigma^2]^T$.
- Evaluate the bias of the ML estimate and compare with parts (a) and (b).
2. Suppose we toss a coin $N$ independent times, where the probability of seeing a heads is $\theta$ (and hence the probability of seeing a tails is $(1-\theta)$), an unknown parameter to be estimated. The outcomes of the coin flips are given by

$$x(n) = \begin{cases} 
1 & \text{if the } n\text{-th outcome is heads} \\
0 & \text{if the } n\text{-th outcome is tails} 
\end{cases}, \quad n = 0, 1, \cdots N - 1$$

(a) Find the density which relates the unknown parameter $\theta$ to our observations $x = [x(0), x(1), \cdots, x(N-1)]^T$, i.e. $p(x; \theta) = \cdots$. *HINT: the probability of a particular $x$ depends on the number of heads and tails in $x$, try to use the fact that $x(n)$ is 1 for heads and 0 for tails to express this.*

(b) Find a sufficient statistic (it exists!) for the estimation of the parameter $\theta$.

(c) Assuming the sufficient statistic you found in (b) is complete, find an MVUE of $\theta$.

(d) Find an ML estimator of $\theta$. *HINT: consider $\ln p(x; \theta)$.*

(e) Find the Cramer-Rao bound for any estimator of $\theta$. 

3. I observe data \( x(n) = A + Bn + n^2 + w(n) \) for \( n = 0, 1, \cdots, N - 1 \), from which I wish to estimate the parameters \( A, B \). I know the following about the noise \( w = [w(0), w(1), \cdots, w(N - 1)]^T \):

\[
E[w] = 0, \quad E[ww^T] = C
\]

(a) If the noise is Gaussian, determine whether an MVUE exists. If so what is it? If not, find a reasonable estimator for \( A \) and \( B \) and explain why it’s reasonable.

(b) If the noise is of unknown density, determine whether an MVUE exists. If so what is it? If not, find a reasonable estimator for \( A \) and \( B \) and explain why it’s reasonable.
4. In a binary communications system, message $m = 0$ and $m = 1$ occur with prior probabilities $p_0 = 1/4$ and $p_1 = 3/4$ respectively. Suppose that we observe $x = m + n$, where $n$ is noise whose distribution is as shown in the Figure (uniform on $[-3/4, 3/4]$). $n$ is also independent of the message $m$.

(a) Find the minimum probability of error detector and compute the associated probability of error.

(b) Suppose the receiver does not know the prior probabilities so it decides to use a maximum likelihood detector. Find the ML detector and the associated probability of error. Is the ML detector unique? Justify your answer.