NAME:

- This exam has 4 questions.
- You will be given the full class time: 75 minutes.
- You may use the 2 course textbooks.
- No calculators are permitted.
- No talking, passing notes, copying (and all other forms of cheating) is permitted.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
1. (30 points) We’re at a fast food chain and we wish to detect whether we’re at Burger King or at Wendy’s. We observe the number of customers $N$ that arrive during a time period, $T$. Someone has already determined the statistics for the two chains, and we know that the probability of $N$ customers arriving during the time period $T$ is given by $p(N; \text{Burger King}) = e^{-k_B T} \frac{(k_B T)^N}{N!}$ at Burger King, while at Wendy’s it’s $p(N; \text{Wendy’s}) = e^{-k_W T} \frac{(k_W T)^N}{N!}$ for $k_B < k_W$. Assume that the prior probability that we’re at Burger King is $p_B$, so that the prior probability that we’re at Wendy’s is $p_W = 1 - p_B$.

(a) Determine the test statistic for deciding whether we’re at Wendy’s or Burger King. Simplify as much as possible.

(b) Defining variables if you need to, express the threshold which will yield the minimum Bayesian risk.

(c) Determine the threshold which will yield the minimum probability of error.

(d) Suppose I wish to fix the probability that we think we’re in Wendy’s but we’re actually in Burger King to 0.1. Write down the expression (in as much detail as possible) that must be satisfied to guarantee this. Indicate what you should solve for but do not solve the equation(s).

(e) Find an expression for the probability of error.
2. (20 points) We wish to determine whether a theorem is correct or not. Rather than proving it ourselves, we try and take a shortcut by asking Nobel prize winners whether they believe the theorem is true or not. Nobel prize winner A correctly guesses whether the theorem is true or not with probability \( p_A \), while nobel prize winner B correctly guesses whether the theorem is true or not with probability \( p_B \). We want to optimally combine the responses from the two winners to guess (i.e. detect) whether the theorem is true or not. The theorem is (a priori) equally likely to be true and false.

(a) Pose this problem as a detection problem: what is the observation, what is the observation space, what are the hypotheses, what is the distribution of the observation under each hypothesis? (HINT: tables may be useful)

(b) Find the decision rule that minimizes the probability of error when \( p_A = 0.7, p_B = 0.6 \). Please also express the simplified decision rule in words.

(c) Find the probability of error for \( p_A = 0.7, p_B = 0.6 \).

(d) In words, how would the decision rule and probability of error change if \( p_A = 0.3 \) and \( p_B = 0.6 \)?
3. (30 points) Consider a feedback system as shown in the Figure below. Here $s(n), n = 0, 1, 2, \cdots, N-1$ is a known signal and $A_i = 0$ under hypothesis $H_0$ (the signal is not present) while it is $A_i = A$ under hypothesis $H_1$ (a signal is present). From the figure, we see that $x(0) = A_is(0) + w(0)$, $r(0) = A_is(0)$ and $x(n) = A_i(s(n) - r(n-1)) + w(n)$ for $n = 1, 2, \cdots, N$. The noise $w(n)$ is zero-mean white Gaussian noise of variance $\sigma^2$.

(a) Write the received signal $x(n)$ directly as a function of $A_i$, $s(n), s(n-1), \cdots s(0)$ and $w(n)$ (that is, eliminate $r(n)$). HINT: write $x(0), x(1), x(2)$ without $r(n)$ and generalize.

(b) Assuming $s = [s(0) s(1) \cdots s(N-1)]^T$ and $A$ and both known, determine the optimal test statistic for deciding if $H_0$ or $H_1$ is true.

(c) Assuming $A$ is unknown, but we know its a priori distribution $p(A)$ describe the new optimal test statistic.

(d) Assuming $A$ is unknown and we do not have a prior distribution on it, and assuming $N = 1$ (that is $s = s(0) \neq 0$ is the only non-zero value of the sent signal), determine the generalized likelihood ratio test statistic. (HINT: if you hit a cubic equation do not solve it and assume a “genie” gives you the solution).

(e) Consider now the system below, where you’ll see that the feedback link has been moved, and $x(0) = A_is(0) + w(0)$ and $x(n) = A_i(s(n) - x(n-1)) + w(n)$ for $n = 1, 2, \cdots, N$. Assume $A$ and $s$ are known. Is the optimal test statistic the same as in question (b)? You do not have to obtain the test statistic, just explain why or why not.
4. (20 points) We observe \( x(n) \) for \( n = 1, 2, \cdots, N \). There are two equally likely hypotheses for the statistical description of \( x(n) \):

\[
\mathcal{H}_0 : x(n) = 1 + w(n) \\
\mathcal{H}_1 : x(n) = -1 + w(n)
\]

where \( w(n) \) is a non-stationary zero-mean white Gaussian sequence \( E[w(n)w(k)] = 0 \) for \( n \neq k \) and \( E[w^2(n)] = \alpha(n) \), a known function of \( n \). That is, the noise \( w = [w(1) w(2) \cdots w(N)]^T \sim \mathcal{N}(0, C_w) \) where \( C_w \) is a diagonal matrix with \( (C_w)_{ii} = \alpha(i) \).

(a) Find the minimum probability of error decision rule and simplify as much as possible.

(b) Find an expression for the probability of error.

(c) Find the asymptotic performance, i.e. \( \lim_{N \to \infty} \Pr(\text{error}) \) when (i) \( \alpha(n) = n\sigma^2 \) and (ii) \( \alpha(n) = n^2\sigma^2 \). (HINT: recall that \( \sum_{i=1}^{\infty} \frac{1}{n} \) diverges to \( \infty \) while \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges to \( \frac{\pi^2}{6} \)).