• This exam has 6 questions.
• You will be given 2 hours.
• You may use the 2 course textbooks but no other aides/notes.
• No calculators are permitted.
• No talking, passing notes, copying (and all other forms of cheating) is permitted.
• Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
• If something has been proven in class or in the book feel free to cite and use the result without a re-derivation.
• Use your time wisely, take shortcuts, use what you know!

Your name: __________________________________________

Your UIN: __________________________________________

Your signature: ______________________________________

The exam has 6 questions, for a total of 100 points.

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1. Suppose $x$ is an unknown parameter and we have 2 observations $y_1$ and $y_2$ with

$$
y_1 = x + n_1$$
$$y_2 = |x| + n_2,$$

where $n_1, n_2$ are independent, identically distributed zero mean Gaussian random variables of variance $\sigma^2$. \textit{HINT: for all these problems you may find it useful to write $|x| = x \text{sgn}(x)$, where $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$.}

(a) (5 points) Find a bound on the estimation error variance of any unbiased estimate of $x$.

(b) (5 points) Does an efficient estimator exist? If so determine it, if not explain why not.

(c) (5 points) Briefly explain or prove why the following is true: if the maximum likelihood estimate $\hat{x}_{ML}(y_1, y_2)$ is nonzero, it has the same sign as $y_1$.

(d) (5 points) Determine the ML estimate of $x$ based on $y_1$ and $y_2$. \textit{HINT: write $\hat{x} = b \text{sgn}(y_1)$ from part (c) and determine the value of $b$ as a function of $y_1$ and $y_2$.}
2. Suppose we are given $y_0$ and $y_1$ as follows:

$$
y_0 = \begin{cases} 
a + n_0, & \text{under hypothesis } H_1 \\
n_0, & \text{under hypothesis } H_0 \end{cases} \quad y_1 = \begin{cases} 
a + b + n_1, & \text{under hypothesis } H_1 \\
n_1, & \text{under hypothesis } H_0 \end{cases}$$

where $n_0$ and $n_1$ are random variables (the noise), and $a$ and $b$ are real constants which we may or may not know (specified in the sub-problems).

(a) (5 points) If $a$ and $b$ are known and the noise $n_0$ and $n_1$ are correlated zero mean, unit variance Gaussian random variables with correlation coefficient $E[n_0 n_1] = 0.5$, describe the detector (no need to find thresholds) that will maximize the probability of detection subject to a fixed probability of false alarm $\alpha$. Find an expression for the probability of detection (no need to simplify if everything is specified).

(b) (5 points) Suppose the noise is the same as in part (a) but that $a$ and $b$ are now under our control, subject to the constraint that $2a^2 + 2ab + b^2 = K$ for some constant $K > 0$. How would you pick $a$ and $b$ to maximize the probability of detection?

(c) (2 points) Suppose that we need to design a Neyman-Pearson detector assuming $n_0$ and $n_1$ are zero-mean, unit variance, uncorrelated Gaussian random variables. Suppose $a$ and $b$ are both unknown, but we know that they are jointly Gaussian, zero mean random variables with covariance matrix $C$. Find the Neyman-Pearson test statistic.

(d) (3 points) From now on assume that hypothesis $H_1$ is true, we know $a$ and want to estimate $b$. Like in part (c) $n_0$ and $n_1$ are zero-mean, unit variance, uncorrelated Gaussian random variables. Can you find a minimum variance unbiased estimator of $b$? If so, find it; if not explain why not.

(e) (5 points) Same assumptions on $n_0$ and $n_1$ as in part (d) but now assume that we also know the prior distribution on $b$, which is uniform on $[-B, B]$ for some known $B$. Write the equation (as fully as possible) that you would need to solve to obtain the Maximum a Posteriori estimate of $b$. 

Points earned: ______________ out of a possible 20 points
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3. Suppose that we are given the following information concerning a pair of random variables $x$ and $y$:

- The marginal probability density for $y$ is given by

$$p(y) = \frac{1}{6}$$

- The MMSE estimate of $x$ based on $y$ is given by

$$\hat{x}_{MMSE} = \begin{cases} 
6, & 0 \leq y < 3 \\
2, & 3 \leq y \leq 6 
\end{cases}$$

(a) (5 points) Determine the Linear Minimum Mean Squared Error (LMMSE) estimator of $x$ based on $y$.

(b) (5 points) Suppose now that we know that the LMMSE estimator of $y$ based on $x$ is given by

$$\hat{y}_{LMMSE}(x) = 5 - \frac{1}{2}x.$$ 

Determine the mean squared error achieved by the LMMSE estimator of $x$ based on $y$. 

Points earned: ___________ out of a possible 10 points
4. Consider a source with four possible outcomes $O_1$, $O_2$, $O_3$, or $O_4$, associated with the signal vectors:

$$S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, S_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, S_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

respectively. The source outcomes $O_1$, $O_2$, $O_3$, and $O_4$ occur with probabilities 0.4, 0.3, 0.2, and 0.1, respectively. A receiver observes the random vector

$$Y = S + N,$$

where $S$ is one of the signal vectors $S_1$, $S_2$, $S_3$ or $S_4$, $Y = [Y_1 Y_2]^T$ is the observed random vector and $N = [N_1 N_2]^T$ is additive noise with distribution

$$p_{N_1,N_2}(n_1,n_2) = \begin{cases} 0.25, & |n_1| < 1, |n_2| < 1 \\ 0, & \text{otherwise} \end{cases}$$

**Hint:** intuition! Don’t get bogged down in math, use pictures/plots whenever possible.

(a) (5 points) Give the expression for $p_{Y_1,Y_2|S_3}(y_1,y_2)$, the conditional probability density function of $Y$ given $S = S_3$.

(b) (5 points) Given $Y$ we wish to determine which of the four outcomes occurred such that the probability of error is minimized. Sketch the decision regions in the plane of the observed values $Y$.

(c) (5 points) What is the corresponding minimum probability of error?
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Points earned: __________ out of a possible 0 points
5. The random variables $X$ and $Y$ are uniformly distributed over the trapezoidal region shown in the figure below, described by the equations

$$p(x, y) = \begin{cases} 
1 & |y| < 1/2, y - 1/2 < x < y + 1/2 \\
0 & \text{else}
\end{cases}$$

(a) (5 points) Find the MMSE of $X$ given $Y$ and the corresponding minimum mean squared error.

(b) (3 points) Find the Linear MMSE of $X$.

(c) (2 points) Define the Quadratic MMSE as the estimator $\hat{X} = aY^2 + bY + c$ where the constants $a, b, c$, are chosen to minimize the mean squared error. Find the quadratic MMSE estimator of $X$ based on $Y$ and the corresponding MMSE. *HINT: think first!*

(d) (5 points) Find the MAP estimate of $X$ based on $Y$. 

Points earned: __________ out of a possible 15 points
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6. Consider the following hypothesis testing problem:

\[ \mathcal{H}_0 \quad \text{coin is fair, } P(\text{head}) = 1/2 \]
\[ \mathcal{H}_1 \quad \text{coin is biased, } P(\text{head}) = p, p \in (1/2, 1] \text{ known} \]

We observe \( N \) independent tosses of the coin and wish to determine whether \( \mathcal{H}_0 \) or \( \mathcal{H}_1 \) occurred.

(a) (5 points) Find the likelihood ratio.

(b) (5 points) What is a sufficient statistic \( T(N) \) for this problem? Simplify the statistic as much as possible.

Now suppose that we have 2 identical looking coins and we know that one is unbiased (probability of heads = probability of tails = 1/2) while the other is biased with probability of heads \( p \), an unknown constant. We do not know which coin is biased and which is unbiased. We do know that the bias \( p \in [0, 1/2) \cup (1/2, 1] \), that is, \( p \neq 1/2 \). The coins are tossed simultaneously and independently to produce an independent sequence of sample \( (X_1, Y_1), (X_2, Y_2), \ldots (X_N, Y_N) \) where each \( X_i, Y_i \) are 0 (tails) or 1 (heads). Our goal is to determine which is the fair coin.

(c) (5 points) Formulate this problem as a binary hypothesis testing problem (specify the hypotheses and the corresponding conditional distributions you will be using).

(d) (5 points) Determine the generalized likelihood ratio test.
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