Problem 1 (2.1)
Solution:

\[
E[\hat{\sigma}^2] = E\left[ \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \right] \\
= \frac{1}{N} \sum_{n=0}^{N-1} E[x^2[n]] \\
= \frac{1}{N} N \sigma^2 \\
= \sigma^2
\]

So this is an unbiased estimator.

\[
Var(\hat{\sigma}^2) = \text{Var}\left( \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \right) \\
= \frac{1}{N^2} \sum_{n=0}^{N-1} \text{Var}(x^2[n]) \\
= \frac{1}{N^2} N \text{Var}(x^2[n]) \\
= \frac{1}{N} \text{Var}(x^2[n])
\]

According to \(x[n]\) is iid, then \(x^2[n]\) is also iid. We know that \(\text{Var}(x^2[n]) = E[x^4[n]] - E[x^2[n]]^2\).

And according to central moment:

\[
E[(x - \mu)^p] = \begin{cases} 
0 & \text{if } p \text{ is odd} \\
\sigma^p (p-1)!! & \text{if } p \text{ is even} 
\end{cases}
\]

\(n!!\) denotes the double factorial, that is the product of every odd number from \(n\) to 1. And \(\mu\) is the mean of \(x\). In this problem, we know that the mean of \(x\) is 0. So we can have \(E[x^4[n]] = 3\sigma^4\). So \(\text{Var}(x^2[n]) = 3\sigma^4 - \sigma^4 = 2\sigma^4\). Then we have

\[
\text{Var}(\hat{\sigma}^2) = \frac{1}{N^2} N 2\sigma^4 = \frac{2\sigma^4}{N}
\]
And $\text{Var}(\hat{\sigma}^2) \to 0$ as $N \to \infty$

Problem 2 (2.3)
Solution:

$$E[\hat{A}] = E\left[\frac{1}{N} \sum_{n=0}^{N-1} x[n]\right]$$
$$= \frac{1}{N} \sum_{n=0}^{N-1} E[x[n]]$$
$$= \frac{1}{N} N A = A$$

$$\text{Var}(\hat{A}) = \text{Var}\left(\frac{1}{N} \sum_{n=0}^{N-1} x[n]\right)$$
$$= \frac{1}{N^2} N \text{Var}(x[n])$$
$$= \frac{1}{N} \sigma^2$$
$$= \sigma^2$$

According to $x[n]$ is iid. Gaussian distributed. So we have $\hat{A} \sim \mathcal{N}(A, \frac{\sigma^2}{N})$.

Problem 3 (2.8)
Solution:
From 2.3, we know that $\hat{A} \sim \mathcal{N}(A, \frac{\sigma^2}{N})$. Then we can have:

$$\lim_{N \to \infty} \Pr\{|\hat{A} - A| > \epsilon\} = \lim_{N \to \infty} \Pr\left\{\frac{|\hat{A} - A|}{\sqrt{\frac{\sigma^2}{N}}} > \frac{\epsilon}{\sqrt{\frac{\sigma^2}{N}}}\right\}$$

According to Q-function, we will have

$$\lim_{N \to \infty} \Pr\left\{\frac{|\hat{A} - A|}{\sqrt{\frac{\sigma^2}{N}}} > \frac{\epsilon}{\sqrt{\frac{\sigma^2}{N}}}\right\} = \lim_{N \to \infty} Q\left(\frac{\epsilon}{\sqrt{\frac{\sigma^2}{N}}}\right)$$
$$= \lim_{N \to \infty} Q\left(\frac{\epsilon \sqrt{N}}{\sigma}\right)$$
$$= 0$$
So \( \hat{A} \) is consistent.

\[
E[\hat{A}] = E\left[\frac{1}{2N} \sum_{n=0}^{N-1} x[n]\right] \\
= \frac{1}{2N} \sum_{n=0}^{N-1} E[x[n]] \\
= \frac{1}{2N} NA \\
= \frac{A}{2}
\]

\[
Var(\hat{A}) = Var\left(\frac{1}{2N} \sum_{n=0}^{N-1} x[n]\right) \\
= \frac{1}{4N^2} \sum_{n=0}^{N-1} Var(x[n]) \\
= \frac{1}{4N^2} N\sigma^2 \\
= \frac{\sigma^2}{4N}
\]

According to \( x[n] \) is iid white Gaussian. So \( \hat{A} \sim \mathcal{N}\left(\frac{A}{2}, \frac{\sigma^2}{4N}\right) \). Then we can have:

\[
\lim_{N \to \infty} Pr\{|\hat{A} - A| > \epsilon\} = \lim_{N \to \infty} Pr\left\{\frac{|\hat{A} - A|}{\sqrt{\frac{\sigma^2}{4N}}} > \frac{\epsilon}{\sqrt{\frac{\sigma^2}{4N}}}\right\}
\]

Also according to Q-function, we can have:

\[
\lim_{N \to \infty} Pr\left\{\frac{|\hat{A} - A|}{\sqrt{\frac{\sigma^2}{4N}}} > \frac{\epsilon}{\sqrt{\frac{\sigma^2}{4N}}}\right\} = \lim_{N \to \infty} Q\left(\frac{\epsilon}{\sqrt{\frac{\sigma^2}{4N}}}\right)
\]

\[
= \lim_{N \to \infty} Q\left(\epsilon\sqrt{\frac{4N}{\sigma^2}}\right) = 0
\]

\( \hat{A} \) is a biased estimator. It is centered at \( \frac{A}{2} \). So \( \hat{A} \) is not consistent.

Problem 4 (2.9)
Solution:

\[
E[\hat{\theta}] = E[(\frac{1}{N} \sum_{n=0}^{N-1} x[n])^2] \\
= Var(\frac{1}{N} \sum_{n=0}^{N-1} x[n]) + E[\frac{1}{N} \sum_{n=0}^{N-1} x[n]^2] \\
= \frac{\sigma^2}{N} + A^2 \\
\neq \theta
\]

So this is a biased estimator. \( E[\hat{\theta}] \to A^2 \) as \( N \to \infty \), this estimator becomes unbiased. This estimator is asymptotically unbiased.