Modular hypercube: a recursive approach to a barrel shifter
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Abstract - Modular Hypercube, which is isomorphic to Barrel Shifter, is presented as an extension of the Hypercube topology. While retaining all the positive features of Hypercube like regularity, isotropic; it is highly symmetric. The network is conceptualized as a recursive structure of virtual modular Hypercubes, which results in simple algorithms for routing and broadcasting. The routing and broadcasting procedures are optimal, as they require a minimum number of hops. The broadcasting procedure, while meeting this criterion also ensures that multiple copies of the broadcast packets are not transmitted to the same node.

Keywords: Barrel Shifter, Hypercube, Modular Hypercube, Routing, Broadcasting.

1 Introduction

Hypercube is widely used topology for computer interconnection networks [2] because of its several attractive features. It is regular, as node degrees are identical. The interconnection pattern is such that it appears to be the same from each node, resulting in isotropic geometry. Partitioning a Hypercube into smaller identical networks is easy to visualize. Efficient procedures for routing and broadcasting for Hypercube structures are available. For most of the applications it is considered to provide a good trade-off between cost, which is measured in terms of node degree, and performance, which is a function of the network diameter.

Several variants of the hypercube network have also been reported to further enhance its performance. These include Banyan Hypercube Networks [5], Hypertree [6], Extended Hypercube [7], Folded Hypercube [8].

The usual modifications are addition of links as in the Folded Hypercube topology, or addition of nodes as in the Extended Hypercube topology, or mapping of another structure on a Hypercube such as Cube-Connected Cycle, Banyan Hypercube, Hypertree etc. Hypertree and Extended Hypercube do not retain the isotropic property of the Hypercube. Banyan Hypercube has an average distance less than that of Hypercube for small network dimensions but increases sharply when the size of the network increases to a large value. In case of Folded Hypercube, the network diameter is half that of the Hypercube but the average distance is not improved much.

Barrel shifter is a well-known topology [1]. While retaining positive properties of a hypercube, the topology doubles its performance. It has been described as a static point-to-point network topology obtained from a ring by adding extra links from each node to
those nodes having a distance equal to an integer power a of 2. In this paper we present
the topology of a Barrel shifter as a Modular Hypercube This results in a better
exploitation of the properties of the Barrel Shifter and leads to efficient routing and
broadcasting procedures.

Section 2 describes Modular Hypercube networks in detail. Their recursive nature
leads to the formation of Virtual Modular Hypercubes, which are conceptualized in
section 3. Some relevant properties of the network are derived in section 4. Finally,
routing and broadcasting methods are developed in section 5.

2 Modular Hypercube Networks

Like hypercube network, MH-Network is also available only in the sizes of nodes \( N = 2^n \).
However in this network the degree of each node is \( (2n-1) \) as compared to degree \( n \) of the
hypercube network. The interconnection pattern is such that its \( n \) adjacent nodes are same
as those in hypercube; and the rest are modular image of these nodes around the node for
which the interconnection is calculated as shown in Figure 1.

The nodes are numbered in the denary notation serially from 0 to \( N-1 \). The node 0
can also be written as the \( N^{th} \) or \( 2^n \)th as per the modular property. Considering the
modular and hyper cubical properties, adjacency function of a node \( x \) in \( n \) dimension
MH-network is defined as: -

\[
A_n(x) = A_n^f(x) \cup A_n^r(x)
\]

where

\[
A_n^f(x) = |x + 2^j| \text{ mod } N \quad j=0,1...(n-1) \ldots (i) \quad \text{is the adjacency function}
\]

\[
\text{defined in the forward direction}
\]

and

\[
A_n^r(x) = |x - 2^j| \text{ mod } N \quad j=0,1...(n-1) \ldots (ii) \quad \text{is the adjacency function}
\]

\[
\text{defined in the reverse direction}
\]

It maybe noted that the operations \(|x + 2^{n-1}| \text{ mod } N\) and \(|x-2^{n-1}| \text{ mod } N\) lead to the
same node. Accordingly, MH acquires mirror image symmetry in the interconnection
pattern about this node, called the node of symmetry with respect to node \( x \).

Definitions:

**Distance:** Distance of a node \( x \) from node \( y \) is defined as the minimum number of links
separating the two. Notation used for distance is \( D_n^x(y) \) and is read as distance of node \( y \)
from node \( x \) in a \( n \)-MH network.

**Difference:** Difference between node \( x \) from node \( y \) is defined as the algebraic difference
between the two. Notation used for distance is \( d_n^x(y) \) and is read as difference of node \( y \)
from node \( x \) in a \( n \)-MH network. Mathematically it is defined as: -

\[
d_n^x(y) = |x - y| = d_n^y(x)
\]

**Node of symmetry:** Node of symmetry with reference to a node \( x \) is the node adjacent to
node \( x \) and with the maximum difference from it. In other terms, this is the node obtained
from both the forward and reverse adjacency function i.e.

\[
|x + 2^{n-1}| \text{ mod } N = |x-2^{n-1}| \text{ mod } N
\]
**Line of symmetry:** The line joining the reference node and node of symmetry in a 2-D plane is known as the line of symmetry.

**Modular shift:** One of the striking properties of MH-network is that if adjacent nodes of node $x$ is known; adjacent nodes of any node $y$ can be calculated by adding the difference $d_{xy}(y)$ to the adjacent nodes in modular fashion. This property can also be easily deduced from the adjacency function and is called the modular shift.

![Figure 1: 4-MH network and its modular properties](image-url)
The figure 1 shows a MH network for \( n = 4 \), called 4-MH. The set of adjacent nodes, (all nodes at unit distance), for a reference node \( x=0 \) is \( \{1,2,4,8,12,14,15\} \). The nodes 1,2,4,8 are referred, as the forward adjacent nodes while nodes 8,12,14,15 are the reverse adjacent nodes. Node 8 is the node of symmetry, which is in both, the forward as well as the reverse adjacent node set. Link connecting the nodes 0 and 8 is the line of symmetry, around which the network has mirror image symmetry. Knowing the set of adjacent nodes of 0, for any other node (say 7) the set can be obtained by adding their difference \( (7) \) to each element, viz. \( \{8, 9, 13, 15, 3, 5, 6\} \).

### 3 Virtual Modular Hypercube Networks

In this section we conceptualize Virtual Modular Hypercube Networks (VMH), which reveal recursive structure of a MH network, help in finding topological properties and lead to simple and efficient methods of routing and broadcasting.

Consider elements of the set of \( 2n-1 \) adjacent nodes of an \( n \)-MH. The difference between every consecutive pair of elements of the set, say \( p \) and \( q \), is a power of 2 \( \{i.e., (p-q) \mod N = 2^k \text{ where, } 0 \leq k \leq (n-2)\} \). Also, the \( k \) forward adjacent terms of \( p \) and \( k \) reverse adjacent terms of \( q \) are included within the subset of \( 2^k \) terms. Further, assume that the node \( q \) is merged with node \( p \) to form a single node, represented as \( p/q \) node. Now with node \( p/q \) as the reference node, a \( k \)-MH may be visualized between the pair of nodes \( p \) and \( q \). This visualized MH is called Virtual MH. For this formation, the two nodes \( p \) and \( q \) are said to contribute the forward and the reverse sub-VMH, respectively.

Thus the every pair of the \( (2n-1) \) adjacent terms of a \( n \)-MH, arranged in ascending order, may be considered to form \( 2^k(n-1) \) VMH of dimensions, \( 2^0, 2^1, \ldots, 2^{k-2}, 2^{k-2}, \ldots, 2^1, 2^0 \) respectively.

The concept of VMH helps in understanding the recursive structure of MH. As every VMH retains properties of a MH, it may have embedded VMH of its own, albeit of smaller dimensions. For a \( k \)-VMH, the maximum and minimum dimensions of an embedded VMH are \( k-2 \) and 0 respectively. Thus, any higher dimensional VMH can be defined recursively in terms of 0-VMH.

To identify a VMH uniquely, we will assign it a level number starting from zero to the top level VMH obtained from a MH under consideration. The level of a VMH is denoted by the subscript, for instance a \( 2^{nd} \) level VMH is denoted as \( \text{VMH}_2 \).

**Example:** In a 5-MH, adjacent nodes to reference node 0, are:
\[ \{1,2,4,8,16,24,28,30,31\} \]
The eight \( \text{VMH}_0 \) networks, formed between the pair of adjacent nodes are:
\[ \{\{1,2\},\{2,4\},\{4,8\},\{8,16\},\{16,24\},\{24,28\},\{28,30\},\{30,31\}\} \]
Consider the VMH between node 8 and 16. The adjacent nodes to these 2 nodes are:
\[ 8 = \{9,10,12\} \text{ (forward direction)} \]
\[ 16 = \{15,14,12\} \text{ (reverse direction)} \]
Now, if these 2 nodes are combined to form a single virtual node 8/16, the adjacency function of MH-network is applicable to the network so formed. Also, \( \{8, 9, 10, 11, 12\} \) is known as **forward sub-VMH** and \( \{16, 15, 14, 13, 12\} \) is known as **reverse sub-VMH**. The four VMH\(_1\) formed with reference node 8/16 are \( \{8,9\}\{10,12\}\{12,14\}\{14,15\} \) and the two VMH\(_2\) formed by reference node 10/12 are \( \{10,11\}\{11,12\} \).
4 Properties

From the above observations and study of the network, we derive few properties of modular hypercube, which will be helpful for developing procedures for routing and broadcasting.

**Proposition 1:**
The distance of all the nodes in forward direction, from a reference node, remains same despite of increasing the dimension of the network i.e. for any two n-MH and k-MH where \( n > k \)
\[
D_{k}^{x}(x+i \mod 2^{k}) = D_{n}^{x}(x+i \mod 2^{n})
\]
where \( i = 0, 1, \ldots, 2^{k-2} \)

**Proposition 2:**
A node residing in a VMH_0 and at distance \( L \) from the reference node is at distance \( L+1 \) from the reference node of the MH-network.

**Proposition 3:**
A node resides in a sub-VMH if and only the reference node of the sub-VMH has the minimum difference with the node.

**Theorem 1:**
For an n-MH, the distance of a node i from reference node x, \( D_{n}^{x}(i) \), can be calculated recursively as:
\[
D_{n}^{x}(i) = 1 + D_{n}^{y}(\min \{ d_{n}^{y}(A_{n}(x)) \})
\]
where \( A_{n}(x) \) is the adjacency function of reference node x
\( d_{n}^{y}(A_{n}(x)) \) is the difference function as defined previously

**Proof:**
For simplicity purpose, the reference node is assumed to be node 0, if proved for node 0, it can be proved for any other node in the similar fashion.
Let, the distance of node i be \( L \) from node 0 in an n-MH.
Now, identify the VMH_0 in which node i reside. The VMH_0 can be written as:
\[
\{y,n_{y1}, n_{y2}, n_{y3}, \ldots, n_{yz}, n_{z2}, n_{z1}, z\}
\]
where \( \{y,n_{y1}, n_{y2}, n_{y3}, \ldots, n_{yz}\} \) is the forward sub-VMH_0 and
\( \{n_{yz}, \ldots, n_{z3}, n_{z2}, n_{z1}, z\} \) is the reverse sub-VMH_0 with \( n_{yz} \) as the node of symmetry.

**Case-I: Node i resides in forward sub-VMH_0**
Let, the distance of node i be \( L^{1} \) from node y.
Referring to Proposition 2, the distance \( L^{1} = L-1 \).
Using Proposition 1, distance of node i from node y is the same as distance of node \( \{d_{n}^{y}(i)\} \) from node 0.
Now,
\( d_{n}^{y}(i) \) can be also written as \( d_{n}^{y}(y) \)
As y is adjacent to node 0 and the closest to node i according to Proposition 3, \( d_{n}^{y}(y) \) can be written as \( \min \{d_{n}^{y}(A_{n}(x))\} \)
Hence, the distance of node i from node 0 is
\[
D_{n}^{x}(i) = L^{1}
= 1 + D_{n}^{x}(\min \{d_{n}^{y}(A_{n}(x))\})
\]
**Case-II: Node i resides in reverse sub-VMH**

According to the properties of MH-networks, the nodes in reverse direction are modular image of the nodes in forward direction i.e. \( n_y_1 \leftrightarrow n_{z_1}, n_y_2 \leftrightarrow n_{z_2}, n_y_3 \leftrightarrow n_{z_3} \) and so on. Hence, node i would have a modular image in forward sub-VMH say \( i^1 \).

From case-I

\[
D_n^x(i^1) = 1 + D_n^x(\min \{ d_n^{i_1}(A_n(x)) \})
\]

Also, as node i and node \( i^1 \) are modular images in the VMH\(_0\),

\[
d_n^i(z) = d_n^{i_1}(y) = \min \{ d_n^i(A_n(x)) \} = \min \{ d_n^{i_1}(A_n(x)) \}
\]

Hence,

\[
D_n^x(i) = 1 + D_n^x(\min \{ d_n^i(A_n(x)) \})
\]

**Theorem 2:**

The number of nodes at a distance ‘d’ in an n-MH-network, with reference to a node, can be recursively calculated as

\[
S_n(d) = 2\left[ \sum S_{k}(d-1) \right]
\]

where \( k = 0,1,...,n-2 \)

and \( S_n(1) = 2^{n-1} \)

**Proof:**

Using Theorem 1, the n-MH-network is divided into top-level k-VMH with reference to node x, where \( k = 0,1,...,n-2 \). (VMH of each dimension in both forward and reverse direction)

Also, the nodes adjacent to node x would be the reference nodes of the respective VMH formed.

According to Proposition 2, the nodes at distance \( d \) in the VMH formed must be at distance \( d+1 \) from node x in the MH-network.

Thus,

\[
S_n(d+1) = S_0(d) + S_1(d) + S_2(d) + ... S_{n-2}(d) \quad \text{in forward direction}
\]

\[
S_n(d+1) = S_0(d) + S_1(d) + S_2(d) + ... S_{n-2}(d) \quad \text{in reverse direction}
\]

Combining the above two gives:

\[
S_n(d+1) = 2 \left[ S_0(d) + S_1(d) + S_2(d) + ... S_{n-2}(d) \right]
\]

Rewriting the above

\[
S_n(d) = 2 \left[ S_0(d-1) + S_1(d-1) + S_2(d-1) + ... S_{n-2}(d-1) \right]
\]

\[
S_n(d) = 2 \left[ \sum S_{k}(d-1) \right]
\]

where \( k = 0,1,...,n-2 \)

As \( S_n(1) \) is equal to \( 2^{n-1} \), the recursive relation is established.

**Theorem 3:**

The diameter of an n-MH-Network is given as:

\[
\text{dia}(n) = \left\lceil n/2 \right\rceil
\]

**Proof:**

The diameter of network is the distance of highest distance node/nodes. These node/nodes are present in the largest VMH only. Thus, the diameter the MH is one more than the diameter of the largest VMH. As \((n-2)\)-VMH is the largest VMH that can be formed in an n-MH, therefore,

\[
\text{dia}(n) = 1 + \text{dia}(n-2)
\]
\[ = 1 + [1 + \text{dia}(n-4)] \]
\[ = 1 + [1 + [1 + \text{dia}(n-6)]] \]
\[ = \sum_{i=1}^{n} 1 + \text{dia}(n-2i) \]
\[ \text{dia}(n) = i + \text{dia}(n-2i) \]

**Case-I: n is an odd number**
\[ \text{dia}(n) = i + \text{dia}(n-2i) \]
\[ = (n-1)/2 + \text{dia}(1) \]
but, \( \text{dia}(1) = 1 \)
Hence, \( \text{dia}(n) = (n-1)/2 + 1 \)
\[ = \lceil n/2 \rceil \]

**Case-II: n is an even number**
\[ \text{dia}(n) = i + \text{dia}(n-2i) \]
\[ = n/2 + \text{dia}(0) \]
but, \( \text{dia}(0) = 0 \)
Hence, \( \text{dia}(n) = n/2 \)
\[ = \lceil n/2 \rceil \]

Therefore, for any \( n \), diameter of a MH-network is given as:
\[ \text{dia}(n) = \lceil n/2 \rceil \]

## 5 Routing and Broadcasting

**Routing over the MH Network**
The routing algorithm being discussed utilizes the VMH property of the network for calculating a shortest default path from source to destination node.

The source node sends the packet to the reference node of the sub-VMH\(_0\) in which the destination node lies. This node is the adjacent node having the least difference from the destination node (as seen from Proposition 3 above). The reference node of sub-VMH\(_0\) routes this packet to the reference node of the sub-VMH\(_1\) embedded in this VMH\(_0\). This way the packet moves into the embedded VMH recursively till it reaches the destination node.

**Algorithm:** The proposed recursive procedure for routing is as follows -
Find the sub-VMH in which the destination node resides using Proposition 3.
Send the message to the reference node sub-VMH obtained. Apply the search recursively till the lowest dimensional (0-VMH) corresponding to the destination node is obtained.

**Example:** In a 5-MH network, routing from source node 0 to node 21 involves the following nodes in the given sequence -
The destination node resides in the VMH\(_0\) with reference node 16/24 (and sub-VMH of reference node 24), i.e. the VMH \{16/24,17,18,19,20,21,22,23\}
Hop 1: 0 \( \rightarrow \) 24
In the next step of recursion, it can be found that the destination node resides in the VMH1 with reference node 20/22. As 21 reside in sub-VMH of both 20 and 22, any one or both can be chosen.

Hop 2: 24 \rightarrow 20 or
   24 \rightarrow 22

From both nodes 20 and 22, destination node 21 is adjacent i.e. it resides in the 0-VMH.

Hop 3: 20 \rightarrow 21 or
   22 \rightarrow 21

The routing path is summarized as:

0 \rightarrow 24 \rightarrow 20 \rightarrow 21 or
0 \rightarrow 24 \rightarrow 22 \rightarrow 21

Broadcasting over the MH Network

According to the properties discussed in the previous section, a MH can be decomposed into VMH0 with respect to the broadcasting node (the reference node). These VMH0 can be further decomposed into still smaller VMH1 and so on. The MH-broadcasting algorithm utilizes this property in the following fashion -

The broadcasting node sends the message packets to all its adjacent nodes. These adjacent nodes would now become the broadcasting nodes for their respective sub-VMH0 in both forward and reverse directions (referred as sub-broadcasting). For each sub-VMH, the procedure is applied recursively till all the sub-VMHs are reduced to single node.

Additional information in message packets

According to the broadcasting algorithm being discussed, each sub-broadcasting node would require to calculate the forward sub-VMH and reverse sub-VMH in which it is required to sub-broadcast.

A VMH can be uniquely defined if the other reference node of the VMH is known. This knowledge is readily available with the node broadcasting/sub-broadcasting to these reference nodes. Hence, the processing overhead can be reduced if this information is sent to the sub-broadcasting node in the message packets.

The additional information for the forward sub-VMH is known as the forward limit and for the reverse sub-VMH is known as the reverse limit.

Special cases

Undefined limits

In a VMH the 2 reference nodes hold the knowledge of its respective half of the network, which is the sub-VMH it owns. Due to which, while sub-broadcasting, it can pass only one of the limits to the node of symmetry. This other limit, which cannot be defined by the reference node of a VMH for the node of symmetry, is known as the node of symmetry’s undefined limit.

But the node of symmetry is self sufficient to calculate the other limit by a simple calculation, which is -

\[ \text{forward limit + reverse limit} = 2^*(\text{node of symmetry}) \]
This equation holds true as the VMH formed on the forward and the reverse direction of the node of symmetry is always of same dimensions making the other reference nodes of the VMHs equidistant (in linear algebra) from it.

**Redundant packets**

It can be observed that the nodes of symmetry in each VMH, will receive duplicate packets, one each from the forward and the reverse path. The redundant packet can be easily avoided by a minimal variation in the algorithm such that in one of the directions, either forward or reverse, the packets are not sent to the node of symmetry.

**Algorithm for broadcasting node**

Step 1 : For each adjacent node process Step 2 and Step 3
Step 2 : Create the message packet with the forward and reverse limit information.
Step 3 : Send the message to the adjacent node.
Step 4 : Exit.

**Algorithm for sub-broadcasting nodes**

Step 1 : On receiving the broadcast packet, read the 2 limits.
Step 2 : If forward limit <> undefined then goto Step 4
Step 3 : Calculate forward limit = 2*(node address) – reverse limit
Step 4 : If reverse limit <> undefined then goto Step 6
Step 5 : Calculate reverse limit = 2*(node address) – forward limit
Step 6 : For each adjacent node in the forward sub-VMH process Step 7 and Step 8
Step 7 : Create the message packet with the forward and reverse limit information.
Step 8 : Send the message to the adjacent node.
Step 9 : For each adjacent node in the reverse sub-VMH process Step 10 and Step 11
Step 10 : Create the message packet with the forward and reverse limit information.
Step 11 : Send the message to the adjacent node.
Step 12 : Exit
**Example:** In a 5-MH network, broadcasting from node 0 involves the following sequence:

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<tr>
<th>Hops</th>
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Should be read as:
- 0 → 8 → 10 → 11
- 0 → 16 → 17
- 0 → 16 → 20 → 21
- etc.
6 Conclusions:

In this paper we have presented a new way of visualizing the well-known network topology of Barrel shifter with respect to static interconnection networks. As an extension of hypercube, it is easy to observe that not only it retains all the positive features of the base topology it enhances them extensively. The concept of virtual modular hypercube gives a useful insight to the topology, which helps in understanding its properties and results in efficient and optimal algorithms for routing and broadcasting.

References