Chapter 4: Entropy rate of a stochastic process

Chapter 4 outline

- Entropy rate
- Stochastic processes and Markov chains

Stochastic processes

- Know entropy of a single as well as multiple RVs. What about the notion of entropy of a more general random process?

Definition: A *stochastic process* $\{X_i\}$ is an indexed sequence of random variables.

Definition: A *discrete-time stochastic process* $\{X_i\}_{i \in \mathcal{I}}$ is one for which we associate the discrete index set $\mathcal{I} = \{1, 2, \ldots\}$ with time.

Entropy: $H(\{X_i\}) = H(X_1) + H(X_2|X_1) + \cdots = \infty$ (often)

- Should probably normalize by n somehow!

Entropy rate

- *Entropy Rate:* The *entropy rate* of a stochastic process $\{X_i\}$ is defined by

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n)$$

  when the limit exists. We can also define an alternative notion:

$$H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n | X_{n-1}, X_{n-2}, \ldots, X_1).$$

- Entropy rate estimates the additional entropy per new sample.
- Gives lower bound on number of code bits per sample.
- If the X_i are not i.i.d the entropy rate limit may not exist.
- X_i i.i.d. random variables: $H(\mathcal{X}) = H(X_i)$

Stationary processes

Definition: A discrete-time stochastic process is said to be *stationary* if the joint distribution of any subset of the sequence of random variables is invariant with respect to shifts in the time index; that is,

$$\Pr\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} = \Pr\{X_{1+l} = x_1, X_{2+l} = x_2, \ldots, X_{n+l} = x_n\}$$

for every n and every shift l and for all $x_1, x_2, \ldots, x_n \in \mathcal{X}$.

Lemma: For a stationary stochastic process, $H(X_n|X_{n-1}, X_{n-2}, \ldots, X_1)$ is nonincreasing in n and has a limit $H'(\mathcal{X})$.

Lemma: Cesáro mean If $a_n \to a$ and $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$, then $b_n \to a$.

Theorem: For a stationary stochastic process, $H(\mathcal{X})$ and $H'(\mathcal{X})$ exist and are equal:

$$H(\mathcal{X}) = H'(\mathcal{X}).$$

Markov chains

- Book has nice results on the entropy rate of Markov chains in Chapter 4. We're skipping it for now, but if you're interested, have a look!