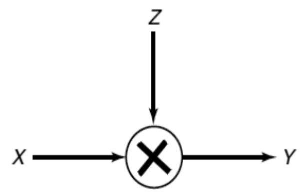
ECE 534: Elements of Information Theory, Fall 2013
Homework 7
Out: October 16, due October 23, 2013.

1. PROBLEM 7.23. *Binary multiplier channel*



(a) Consider the channel $Y = XZ$, where X and Z are independent binary random variables that take on values 0 and 1. Z is Bernoulli(α) [i.e., $P(Z = 1) = \alpha$]. Find the capacity of this channel and the maximizing distribution on X.

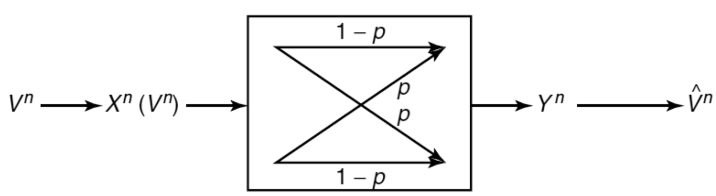
(b) Now suppose that the receiver can observe Z as well as Y. What is the capacity?

2. PROBLEM 7.28. *Choice of channels.* Find the capacity C of the union of two channels $(X_1, p_1(y_1|x_1), Y_1)$ and $(X_2, p2(y_2|x_2), Y_2)$, where at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume that the output alphabets are distinct and do not intersect.

(a) Show that $2^C = 2^{C_1} + 2^{C_2}$. Thus, 2^C is the effective alphabet size of a channel with capacity C.

(b) Compare with Problem 2.10 where $2^H = 2^{H_1} + 2^{H_2}$, and interpret part (a) in terms of the effective number of noise-free symbols.

3. PROBLEM 7.31. *Source and channel.* We wish to encode a Bernoulli(α) process $V_1, V_2, ...$ for transmission over a binary symmetric channel with crossover probability p.



Find conditions on α and p so that the probability of error $P(\hat{V}^n \neq V^n)$ can be made to go to zero as $n \to \infty$.

4. PROBLEM 8.1. *Differential entropy.* Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

   (a) The exponential density, $f(x) = \lambda e^{-\lambda x}, x \geq 0.$

   (b) The Laplace density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$**.**

   (c) The sum of X_1 **and** X_2**, where** X_1 **and** X_2 **are independent normal random variables with means** μ_i **and variances** σ_i^2**,** $i = 1, 2.$

5. PROBLEM 8.3. *Uniformly distributed noise.* Let the input random variable X to a channel be uniformly distributed over the interval $-1/2 \leq x \leq 1/2$. Let the output of the channel be $Y = X + Z$, where the noise random variable is uniformly distributed over the interval $-a/2 \leq z \leq +a/2$.

   (a) Find $I(X;Y)$ **as a function of a.**

   (b) For $a = 1$ **find the capacity of the channel when the input X is peak-limited; that is, the range of X is limited to** $-1/2 \leq x \leq 1/2$**. What probability distribution on X maximizes the mutual information I (X; Y)?**

   (c) (Optional – from textbook) Find the capacity of the channel for all values of a, again assuming that the range of X is limited to $-1/2 \leq x \leq 1/2$**.**