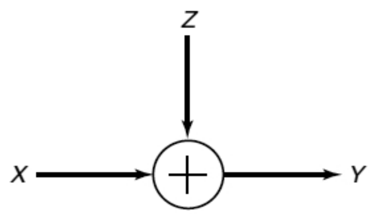
ECE 534: Elements of Information Theory, Fall 2013
Homework 6
Out: October 9, due October 16, 2013.

1. Problem 7.2. *Additive noise channel.* Find the channel capacity of the following discrete memoryless channel:



Where $PrZ = 0 = PrZ = a = \frac{1}{2}$. The alphabet for x is $\mathbf{X} = 0, 1$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

2. Problem 7.7. *Cascade of binary symmetric channels.* Show that a cascade of n identical independent binary symmetric channels,

$X_0 \rightarrow \boxed{\text{BSC}} \rightarrow X_1 \rightarrow ... \rightarrow X_{n-1} \rightarrow \boxed{\text{BSC}} \rightarrow X_n,$

each with raw error probability p, is equivalent to a single BSC with error probability $\frac{1}{2}(1-(1-2p)^n)$ and hence that $lim_{n\to\infty} I(X_0; X_n) = 0$ if $p \neq 0, 1$. No encoding or decoding takes place at the intermediate terminals $X_1, ..., X_{n-1}$. Thus, the capacity of the cascade tends to zero.

3. Problem 7.10. *Zero-error capacity.* A channel with alphabet $0, 1, 2, 3, 4$ has transition probabilities of the form

$$p(y|x) = \begin{cases} 1/2 & \text{if } y = x \pm 1 mod 5 \\ 0 & \text{otherwise} \end{cases}$$

- Compute the capacity of this channel in bits.
- The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-error capacity of this pentagonal channel is at least 1 bit (transmit 0 or 1 with probability 0.5). Find a block code that shows that the zero error capacity is greater than 1 bit. Can you estimate the exact value of the zero-error capacity? (Hint: Consider codes of length 2 for this channel.) The zero-error capacity of this channel was finally found by Lovasz [365].

4. Problem 7.13. *Erasures and errors in a binary channel.* Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ϵ and the probability of erasure be α, so the channel is as in Figure 1.

- Find the capacity of this channel.
- Specialize to the case of the binary symmetric channel ($\alpha = 0$).
- Specialize to the case of the binary erasure channel ($\epsilon = 0$).

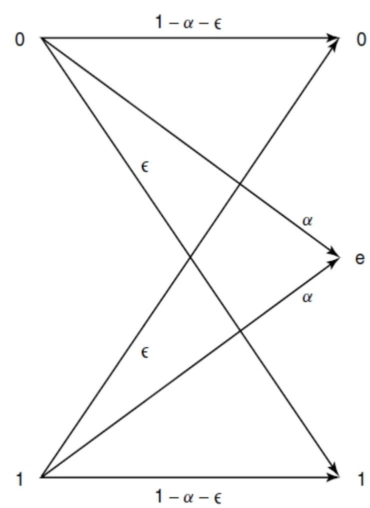


Figure 1: Figure for problem 7.13

5. PROBLEM 7.19. *Capacity of the carrier pigeon channel.* Consider a commander of an army besieged in a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one letter (8 bits), that pigeons are released once every 5 minutes, and that each pigeon takes exactly 3 minutes to reach its destination.

   Set up an appropriate model for the channel in each of the cases, and indicate how to go about finding the capacity.

   (a) Assuming that all the pigeons reach safely, what is the capacity of this link in bits/hour?

   (b) Now assume that the enemies try to shoot down the pigeons and that they manage to hit a fraction α of them. Since the pigeons are sent at a constant rate, the receiver knows when the pigeons are missing. What is the capacity of this link?

   (c) Now assume that the enemy is more cunning and that every time they shoot down a pigeon, they send out a dummy pigeon carrying a random letter (chosen uniformly from all 8-bit letters). What is the capacity of this link in bits/hour?

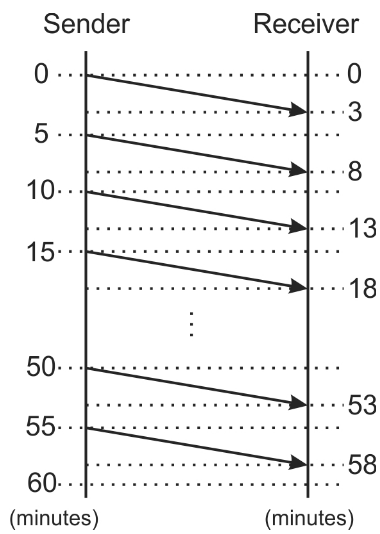


Figure 2: Problem 7.19a