ECE 534: Elements of Information Theory, Fall 2013
Homework 4
Out: September 18, due September 25, 2013.

Problem 3.9 Let $X_1, X_2, \cdots$ be independent, identically distributed random variables drawn according to the probability mass function $p(x)$, $x \in \{1, 2, \cdots m\}$. Thus, $p(X_1, x_2, \cdots, x_n) = \prod_{i=1}^n p(x_i)$. We know that $-\frac{1}{n} \log p(X_1, X_2, \cdots X_n) \to H(X)$ in probability. Let $q(x_1, x_2, \cdots x_n) = \prod_{i=1}^n q(x_i)$, where q is another probability mass function on $\{1, 2, \cdots m\}$.

1. Evaluate $\lim -\frac{1}{n} \log q(X_1, X_2, \cdots X_n)$, where $X_1, X_2, \cdots$ are i.i.d. $\sim p(x)$.
2. Now evaluate the limit of the log likelihood ratio $\frac{1}{n} \log \frac{q(X_1, X_2, \cdots X_n)}{p(X_1, X_2, \cdots X_n)}$ when $X_1, X_2, \cdots$ are i.i.d. $\sim p(x)$. Thus, the odds favoring q are exponentially small when p is true.

Problem 4.8 A discrete memoryless source has the alphabet $\{1, 2\}$, where the symbol 1 has duration 1 and the symbol 2 has duration 2. The probabilities of 1 and 2 are p_1 and p_2, respectively. Find the value of p_1 that maximizes the source entropy per unit time $H(\mathcal{X}) = \frac{H(X)}{E[T]}$. What is the maximum value of $H(\mathcal{X})$?

Problem 4.11 Let $\cdots, X_{-1}, X_0, X_1, \cdots$ be a stationary (not necessarily Markov) stochastic process. Which of the following statements are true? Prove or provide a counterexample.

1. $H(X_n|X_0) = H(X_{-n}|X_0)$
2. $H(X_n|X_0) \geq H(X_{n-1}|X_0)$
3. $H(X_n|X_1, X_2, \cdots, X_{n-1}, X_{n+1})$ is nonincreasing in n
4. $H(X_n|X_1, \cdots, X_{n-1}, X_{n+1}, \cdots, X_{2n})$ is nonincreasing in n

Problem 5.2 (How many fingers does a Martian have?) Let

$$S = \left(\begin{array}{c} S_1, \cdots S_m \\ p_1, \cdots p_m \end{array} \right).$$

The S_i's are encoded into strings from a D-symbol output alphabet in a uniquely decodable manner. If $m = 6$, and the codeword lengths are $(l_1, l_2, \cdots l_6) = (1, 1, 2, 3, 2, 3)$ find a good lower bound on D. You may wish to explain the title of the problem.