ECE 534: Elements of Information Theory, Fall 2013
Homework 11
Out: November 20, Due: November 27, 2013

1. PROBLEM 10.14. *Rate distortion for two independent sources.* Can one compress two independent sources simultaneously better than by compressing the sources individually? The following problem addresses this question. Let $\{X_i\}$ be i.i.d. $\sim p(x)$ with distortion $d(x, \hat{x})$ and rate distortion function $R_X(D)$. Similarly, let $\{Y_i\}$ be i.i.d. $\sim p(y)$ with distortion $d(y, \hat{y})$ and rate distortion function $R_Y(D)$. Suppose we now wish to describe the process $\{(X_i, Y_i)\}$ subject to distortions $Ed(X, \hat{X}) \leq D1$ and $Ed(Y, \hat{Y}) \leq D2$. Thus, a rate $R_{X,Y}(D1, D2)$ is sufficient, where

$$R_{X,Y}(D_1, D_2) = \min_{p(\hat{x},\hat{y}|x,y):Ed(X,\hat{X})\leq D_1, Ed(Y,\hat{Y})\leq D_2} I(X, Y; \hat{X}, \hat{Y})$$

Now suppose that the $\{X_i\}$ process and the $\{Y_i\}$ process are independent of each other.

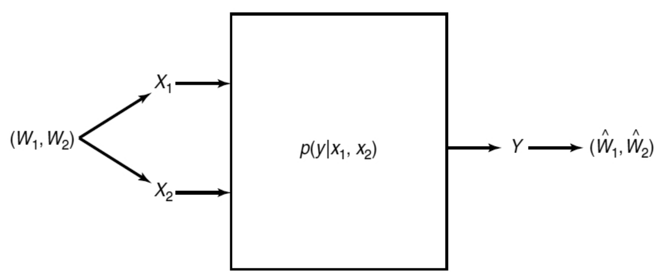
(a) Show that:

$$R_{X,Y}(D_1, D_2) \geq R_X(D_1) + R_Y(D_2)$$

(b) Does equality hold?

Now answer the question: **Can one compress two independent sources simultaneously better than by compressing the sources individually?**

2. PROBLEM 15.1. *Cooperative capacity of a multiple-access channel*



(a) Suppose that X_1 and X_2 have access to both indices $W_1 \in \{1, 2^{nR}\}$, $W_2 \in \{1, 2^{nR_2}\}$. Thus, the codewords $\mathbf{X}_1(W_1, W_2)$, $\mathbf{X}_2(W_1, W_2)$ depend on both indices. Find the capacity region.

(b) Evaluate this region for the binary erasure multiple access channel $Y = X_1 + X_2$, $X_i \in \{0, 1\}$. Compare to the noncooperative region.

3. PROBLEM 15.2. *Capacity of multiple-access channels.* Find the capacity region for each of the following multiple-access channels:

   (a) Additive modulo 2 multiple-access channel. $X_1 \in \{0,1\}$**,** $X_2 \in \{0,1\}$**,** $Y = X_1 \oplus X_2$**.**

   (b) Multiplicative multiple-access channel. $X_1 \in \{-1,1\}$**,** $X_2 \in \{-1,1\}$**,** $Y = X_1 \cdot X_2$**.**

4. PROBLEM 15.6. *Unusual multiple-access channel.* Consider the following multiple-access channel: $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0,1\}$. If $(X_1, X_2) = (0,0)$, then $Y = 0$. If $(X_1, X_2) = (0,1)$, then $Y = 1$. If $(X_1, X_2) = (1,0)$, then $Y = 1$. If $(X_1, X_2) = (1,1)$, then $Y = 0$ with probability $\frac{1}{2}$ and $Y = 1$ with probability $\frac{1}{2}$.

   (a) Show that the rate pairs $(1,0)$ **and** $(0,1)$ **are achievable**.

   (b) Show that for any non-degenerate distribution $p(x_1)p(x_2)$**, we have** $I(X_1, X_2; Y) < 1$**.**

   (c) Argue that there are points in the capacity region of this multiple-access channel that can only be achieved by timesharing; that is, there exist achievable rate pairs $(R1, R2)$ **that lie in the capacity region for the channel but not in the region defined by:**

$$R1 \le I(X_1; Y|X_2), \tag{1}$$
$$R2 \le I(X_2; Y|X_1), \tag{2}$$
$$R1 + R2 \le I(X_1, X_2; Y) \tag{3}$$