Bi-variate Gaussian RVs (2-D Gaussian)

two random variables

$$X \sim N(\underset{E[X]}{\mu_X}, \underset{Var(X)}{\sigma_X^2}) \text{ then } f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

(X, Y) are said to be "jointly Gaussian) if their pdf is of the following form:

correlation coefficient btw. X and Y.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\underbrace{\begin{bmatrix} \overset{E[X]}{\mu_1} \\ \underset{E[Y]}{\mu_2} \end{bmatrix}}_{\text{mean vector}}, \begin{bmatrix} \overset{Var(X)}{\sigma_1^2} & \overset{cov(X,Y)}{\rho\sigma_1\sigma_2} \\ \underset{\substack{cov(Y,X) \\ = cov(X,Y)}}{\rho\sigma_1\sigma_2} & \underset{Var(Y)}{\sigma_2^2} \end{bmatrix} \right)$$

Continuous RVs with pdf:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}{2(1-\rho^2)}}$$

$$\begin{bmatrix} \mu_1 = E[X] \\ \mu_2 = E[Y] \end{bmatrix} = E\begin{bmatrix} X \\ Y \end{bmatrix}$$

$$Cov(X,Y) = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix}$$

Need $-1 \le \rho \le +1$, $\sigma_1 > 0$, $\sigma_2 > 0$.

Properties:

1) Marginals of $f_{XY}(x,y)$ are $f_X(x) \rightarrow$ Gaussian $N(\mu_X, \sigma_1^2)$

$f_Y(y) \rightarrow$ Gaussian $N(\mu_2, \sigma_2^2)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

2) The pdf of Y given X is also Gaussian!

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\tilde{\sigma}_2^2}} e^{-\frac{(y-\tilde{\mu}_2)^2}{2\tilde{\sigma}_2^2}}$$

where $\tilde{\mu}_2 = \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x-\mu_1)$, $\tilde{\sigma}_2^2 = \sigma_2^2(1-\rho^2)$

$\Downarrow$

Get it by $f_{XY}(x,y) = f_X(x) f_{Y|X}(y|x)$.

3) The correlation coefficient is $\rho_{XY} = \rho$.

4) X, Y uncorrelated $\underset{\text{iff.}}{\Longleftrightarrow}$ X, Y independent $(f_{XY}(x,y) = f_X(x) f_Y(y))$

NOT TRUE IN GENERAL, HOLDS FOR GAUSSIAN RVs!!

EX: Textbook 4.11.6

Under what conditions on the constants a, b, c, d is

$$f_{XY}(x,y) = d\,e^{-(a^2x^2 + bxy + c^2y^2)} \quad (*)$$

jointly Gaussian?

Sol: $d = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$, $a^2 = \dfrac{1}{2\sigma_1^2(1-\rho^2)}$

$c^2 = \dfrac{1}{2\sigma_2^2(1-\rho^2)}$, $b = \dfrac{+\rho}{\sigma_1\sigma_2(1-\rho^2)}$

Then the above $(*)$ could be a zero-mean Gaussian if a,b,c,d satisfy:

$$\underbrace{-1 \le \rho \le +1, \quad \sigma_1 > 0, \quad \sigma_2 > 0.}_{\text{What do these conditions imply about } a,b,c,d?}$$

Solve for σ_1, σ_2: $\sigma_1 = \dfrac{1}{a\sqrt{2(1-\rho^2)}}$, $\sigma_2 = \dfrac{1}{c\sqrt{2(1-\rho^2)}}$

Plug σ_1, σ_2 into equation for b: $b = \dfrac{\rho\, a\, \cancel{2(1-\rho^2)}\, c}{\cancel{(1-\rho^2)}\cancel{\sqrt{2(1-\rho^2)}}\cancel{\sqrt{2(1-\rho^2)}}}$

$= 2\rho a c$

or $\rho = \dfrac{b}{2ac}$ conditions.

Since we need $|\rho| \le 1$, we need $\boxed{|b| \le 2ac.}$

Then, for any a,b,c that satisfy we can find $\boxed{d^2 = a^2c^2 - b^2/4.}$