Gaussian random vectors

Def: An n-dimensional ~~RV~~ random vector $\underline{X}$ is called a Gaussian random vector (or $[X_1, X_2, \dots, X_n]^T = \underline{X}$ is called "jointly Gaussian") if they have the following joint pdf:

($\underline{X}$: $n \times 1$ vector)

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\left[\det(C_{\underline{X}})\right]^{1/2}} \exp\left(-\frac{1}{2}(\underline{x}-\mu_{\underline{x}})^T C_{\underline{x}}^{-1}(\underline{x}-\mu_{\underline{x}})\right) \quad (*)$$

denoted as $\underline{X} \sim \mathcal{N}(\mu_{\underline{x}}, C_{\underline{x}})$

with $E[\underline{X}] = \mu_{\underline{x}}$ $(n \times 1)$

$C_{\underline{x}}$ = covariance matrix $= E\left[(\underline{X}-\mu_{\underline{x}})(\underline{X}-\mu_{\underline{x}})^T\right]$ $(n \times n)$

with $\det(C_{\underline{x}}) > 0$.

(Recall) 1-D Gaussian random variable has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

where $E[X] = \mu_X$, $Var(X) = \sigma_X^2$

Let's see if we can get 2-D Gaussian random variable pdf in this form.

(Recall) 2-D Gaussian random vector $\bar{X} = \begin{bmatrix} X \\ Y \end{bmatrix}$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{-\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} \cancel{+} \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \right)$$

From the vector formulation

$$\bar{X} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \mu_{\bar{X}} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad C_{\underline{X}} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Then, plugging into $(*)$ for $n=2$.

$$\boxed{C_{\underline{X}}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}}$$

$$\det(C_{\underline{X}}) = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$$

So, $$\frac{1}{(2\pi)^{2/2}\,|\det C_X|^{1/2}} = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

Then $$(\underline{X}-\mu_{\underline{X}})^T C_{\underline{X}}^{-1}(\underline{X}-\mu_{\underline{X}}) = \overset{1\times 2}{\begin{bmatrix} X-\mu_1 & Y-\mu_2 \end{bmatrix}} \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \overset{2\times 2}{\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}} \overset{2\times 1}{\begin{bmatrix} X-\mu_1 \\ Y-\mu_2 \end{bmatrix}}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \overset{1\times 2}{\begin{bmatrix} X-\mu_1 & Y-\mu_2 \end{bmatrix}} \overset{2\times 1}{\begin{bmatrix} \sigma_2^2(X-\mu_1) - \rho\sigma_1\sigma_2(Y-\mu_2) \\ -\rho\sigma_1\sigma_2(X-\mu_1) + \sigma_1^2(Y-\mu_2) \end{bmatrix}}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \Big[\sigma_2^2(X-\mu_1)^2 - \rho\sigma_1\sigma_2(Y-\mu_2)(X-\mu_1) - \rho\sigma_1\sigma_2(X-\mu_1)(Y-\mu_2) + \sigma_1^2(Y-\mu_2)^2 \Big].$$

$$= \frac{\left(\frac{X-\mu_1}{\sigma_1}\right)^2}{(1-\rho^2)} - \frac{2(Y-\mu_2)(X-\mu_1)}{\sigma_1\sigma_2(1-\rho^2)} + \frac{\left(\frac{Y-\mu_2}{\sigma_2}\right)^2}{(1-\rho^2)}$$

<u>Thm</u>: A Gaussian random vector $\underline{X}$ has independent components $\Longleftrightarrow$ $C_{\underline{X}}$ is diagonal.

<u>Why?</u>: If $\underline{X}$ is jointly Gaussian then the individual components $X_1, X_2, \ldots, X_n$ are all Gaussian (marginals are 1-D Gaussian)

If $C_{\underline{X}}$ is diagonal, then $\underbrace{E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})]}_{(C_{\underline{X}})_{ij}} = 0$

$\Longleftrightarrow$ X_i and X_j are uncorrelated $\Longleftrightarrow$ X_i and X_j are independent since X_i and X_j are <u>Gaussian</u>.

Note that all marginals $X_i \sim \mathcal{N}(\mu_{X_i}, \sigma_{X_i}^2)$

<u>Proof</u>: $\underline{X}$ independent components $\Rightarrow$ C_X diagonal.

$\Rightarrow \mathrm{Cov}(X_i, X_j) = 0$ since X_i, X_j Gaussian $\blacksquare$.

$C_{\underline{X}}$ diagonal $\Rightarrow$ $\underline{X}$ independent

$$C_{\underline{X}} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \Rightarrow C_{\underline{X}}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \frac{1}{\sigma_2^2} \ddots & \\ 0 & & \frac{1}{\sigma_n^2} \end{bmatrix}$$

Then $\det(C_X) = \prod_{i=1}^{n} \sigma_i^2$

Hence, $\left(f_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \prod_{i=1} f_{X_i}(x_i) \right)$ Def. of independ. (4)

$$(\underline{X} - \mu_{\underline{X}})^T C_{\underline{X}}^{-1} (\underline{X} - \mu_{\underline{X}}) = \sum_{i=1}^{n} \frac{(X - \mu_i)^2}{\sigma_i^2}.$$

diagonal! work it out!

Hence,

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} \left(\prod_{i=1}^{n} \sigma_i^2 \right)^{1/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \left(\frac{X - \mu_i}{\sigma_i} \right)^2 \right)$$

(TYPO in 5.69 in book) 1/2 is missing.

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} \exp\left(-\frac{1}{2} \left(\frac{X - \mu_i}{\sigma_i} \right)^2 \right)$$

$$= \prod_{i=1}^{n} \underbrace{f_{X_i}(x_i)}_{\sim N(\mu_{X_i}, \sigma_{X_i}^2)}$$