Vectors of random variables

$\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ is a random vector with pdf $f_{\bar{X}}(\bar{x})$

↳ scalar function of vector argument

$\mu_{\bar{X}} = E[\bar{X}]$ mean of X is $(n \times 1)$

$$= \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

$R_{\bar{X}} = E[\bar{X}\bar{X}^T]$ is correlation matrix is $(n \times n)$

$$= \begin{bmatrix} E[X_1^2] & E[X_1 X_2] & E[X_1 X_3] \\ E[X_2 X_1] & E[X_2^2] & E[X_2 X_3] \\ E[X_3 X_1] & E[X_3 X_2] & E[X_3^2] \end{bmatrix}$$

(for $\bar{X}$ is (3×1))

$C_{\bar{X}} = E[(\bar{X} - \mu_{\bar{X}})(\bar{X} - \mu_{\bar{X}})^T]$ is covariance matrix is $(n \times n)$.

$$= \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) \\ Cov(X_2, X_1) & Var(X_2) & Cov(X_2, X_3) \\ Cov(X_3, X_1) & Cov(X_3, X_2) & Var(X_3) \end{bmatrix}$$

Thm: $C_{\bar{X}} = R_{\bar{X}} - \mu_{\bar{X}} \mu_{\bar{X}}^T$

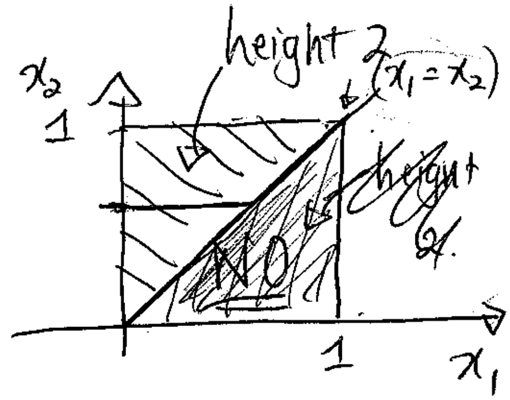
EX: Find $E[X]$, C_X, R_X for the 2-D random vector X with PDF

(dropping the $\bar{X}$ bar notation for my own convenience) $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

(x_1, x_2)

$$f_X(x) = \begin{cases} 2 & 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{else} \end{cases}$$

$[X_1\ X_2]$



$$E[X_i] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i \, f_{X_1 X_2}(x_1, x_2) \, dx_1 \, dx_2$$

$i \in \{1, 2\}$

$$= \int_0^1 \underbrace{\int_0^{x_2} 2 x_i \, dx_1}_{} \, dx_2 = \begin{cases} 1/3 & i = 1 \quad E[X_1] = 1/3 \\ 2/3 & i = 2. \quad E[X_2] = 2/3. \end{cases}$$

So $\mu_{\bar{X}} = E[X] = [1/3 \quad 2/3]^T$

Let's get R_X, then $C_X = R_X - \mu_X \mu_X^T$

$$R_{\bar{X}} = \begin{bmatrix} E[X_1^2] & E[X_1 X_2] \\ E[X_1 X_2] & E[X_2^2] \end{bmatrix}.$$

$$E[X_1^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1^2 f_{X_1 X_2}(x_1, x_2) \, dx_1 \, dx_2 = \int_0^1 \int_0^{x_2} 2 x_1^2 \, dx_1 \, dx_2 = 1/6.$$

$$E[X_2^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_2^2 f_{X_1 X_2}(x_1, x_2) \, dx_1 \, dx_2 = \int_0^1 \int_0^{x_2} 2 x_2^2 \, dx_1 \, dx_2 = 1/2$$

$$E[X_1 X_2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1 X_2}(x_1, x_2) \, dx_1 \, dx_2 = \int_0^1 \int_0^{x_2} 2 x_1 x_2 \, dx_1 \, dx_2 = 1/4$$

Thus, $R_{\bar{X}} = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$ symmetric as expected!

$$C_{\bar{X}} = R_{\bar{X}} - \mu_{\bar{X}}\mu_{\bar{X}}^T = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix}$$

Vector cross-correlation

The cross-correlation of random vectors X with n components $(n \times 1)$ and random vector Y with m components $(m \times 1)$ is a matrix R_{XY} whose (i,j) th element is

$$R_{XY}(i,j) = E[X_i Y_j] \qquad \underbrace{R_{XY} = E[XY^T]}_{n \times m \text{ matrix}}.$$

Vector cross-covariance

$$C_{XY}(i,j) = Cov(X_i, Y_j) = E[(X_i - \mu_{X_i})(Y_j - \mu_{Y_j})]. \qquad \underbrace{C_{XY} = E[(X-\mu_X)(Y-\mu_Y)^T]}_{n \times m \text{ matrix.}}$$

Thm. 5.13 + 5.14. If X is $(n \times 1)$ Random vector with μ_X, R_X, C_X and

$$Y = AX + b$$

A : known $m \times n$ matrix
b : known $m \times 1$ vector

Then

$$\mu_Y = A\mu_X + b.$$

$$R_Y = AR_XA^T + (A\mu_X)b^T + b(A\mu_X)^T + bb^T$$

$$C_Y = AC_XA^T$$

$$R_{XY} = R_XA^T + \mu_X b^T$$

$$C_{XY} = C_XA^T$$

Proof: (just showing μ_Y, C_Y, R_{XY}, can do rest similarly)

$$\mu_Y = E[AX+b] \overset{\text{linearity}}{=} E[AX] + E[b] \overset{\substack{A \text{ is constant} \\ + \text{linearity}}}{=} AE[X] + b. = A\mu_X + b.$$

(Then $Y - \mu_Y = (AX+b) - (A\mu_X + b) = A(X - \mu_X)$)

$$C_Y = E[(Y-\mu_Y)(Y-\mu_Y)^T] = E[(A(X-\mu_X))(A(X-\mu_X))^T].$$

$$= E[A(X-\mu_X)(X-\mu_X)^TA^T] = A\underbrace{E[(X-\mu_X)(X-\mu_X)^T]}_{C_X}A^T$$

$$= AC_XA^T$$

$$R_{XY} = E[XY^T] = E[X(AX+b)^T] = E[X(X^TA^T + b^T)].$$

$$= E[XX^T]A^T + E[X]b^T = R_XA^T + \mu_X b^T$$

Book problem 5.4.1: (I think has a typo)

The n components X_i of a random vector X have:

$$E[X_i]=0, \quad Var(X_i)=\sigma_i^2. \qquad (i=1,2,\dots,n).$$

What is the covariance matrix C_X?

Knowing only this, I can conclude the following:

$$C_X = \begin{bmatrix} \boxed{Var(X_1)} & & & \\ & Var(X_2) & & \\ & & \ddots & \\ & & & Var(X_n) \end{bmatrix} \quad n \times n.$$

$$= \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$

cannot fill these in without further information!

I think they meant components X_i are all independent.

If X_i, X_j for $i \neq j$ are independent then $Cov(X_i, X_j)=0$

Thus

$$C_X = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \sigma_n^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & 0 \\ & 0 & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$