$$\mathrm{Var}[X] \triangleq E\left[(X-\mu_X)^2\right] = \sum_{x \in S_X} (x-\mu_X)^2 P_X(x)$$

X is R.V. with p.m.f. $P_X(x)$, and range S_X $\qquad \mu_X \triangleq E[X] = \sum_{x \in S_X} x \cdot P_X(x)$

Var[X] if X is Bernouilli p. $\rightarrow$ Var[Bernouilli(p)] = p(1-p).

<u>Ex</u>: $\mathrm{Var}[X+5] = \mathrm{Var}[X] \qquad (\mathrm{Var}[X + \underset{\text{constant}}{K}] = \mathrm{var}[X])$

$$\mathrm{Var}[X+5] = \sum_{x \in S_X} \left((x+5) - E[X+5]\right)^2 P_X(x) = \cancel{E[(X-\mu_X}$$

$E[5] = 5$

$$= E\left[(X+5-E[X+5])^2\right].$$

$$= \sum_{x \in S_X} (x+5-E[X]-5)^2 P_X(x).$$

$$= \sum_{x \in S_X} (x - E[X])^2 P_X(x) = \mathrm{Var}[X].$$

<u>Conditional Probability Mass Functions</u>

Recall, given events A, B $\quad P[A|B] = \dfrac{P[AB]}{P[B]} = \dfrac{P[B|A]P[A]}{P[B]}$

if $P[B] > 0$.

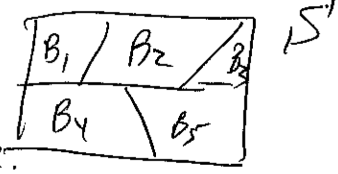
For a R.V. X,

$$P[(X=x_0)|B] = \frac{P[(X=x_0), B]}{P[B]} \text{ for an event } B \text{ with } P[B] > 0.$$

Then, the <u>conditional pmf</u> is given by.

$$P_{X|B}(x_0) \triangleq \frac{P[X=x_0, B]}{P[B]}.$$

Given an event space $\{B_1, B_2, \ldots, B_n\}$
→ collectively exhaustive.
→ mutually exclusive.



$$P_X(x) = \sum_{i=1}^{n} P_{X|B_i}(x)\, P[B_i].$$

EX: X has CDF $F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \le x < 5 \\ 0.8 & 5 \le x < 7 \\ 1 & 7 \le x \end{cases}$

$F_X(x) = P[X \le x]$.

- What is S_X? $S_X = \{-3, 7, 5\}$. $P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \end{cases}$

- If $B = \{X > 0\}$, find $P[B]$:

$P[B] = P[\{X=5\} \cup \{X=7\}] = P[X=5] + P[X=7] = 0.6.$

- Find $P_{X|B}(x)$:

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{P(B)} & x \in B \\ 0 & \text{else} \end{cases} = \begin{cases} \dfrac{P_X(5)}{0.6} & x = 5 \\ \dfrac{P_X(7)}{0.6} & x = 7 \\ 0 & x = -3. \end{cases} = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & \text{else.} \end{cases}$$

- Find $E[X|B]$, $E[X^2|B]$, $E[X|B^c]$.

$$E[X|B] = \sum_{x \in S_{X|B}} x \cdot P_{X|B}(x) = 5 \cdot P_{X|B}(5) + 7 \cdot P_{X|B}(7) = 5 * 2/3 + 7 * 1/3 = 17/3$$

$S_{X|B} = \{5, 7\}$.

$$E[X^2|B] = \sum_{x \in S_{X|B}} x^2 P_{X|B}(x) = 25 * 2/3 + 49 * 1/3 = 33.$$

$B^c = \{X = -3\}$., $\cancel{P[X=-3] = P[B^c] = \begin{cases} 1 & x=-3 \\ 0 & \text{else} \end{cases}}$

$$P_{X|B^c}(x) = \begin{cases} \frac{P_X(x)}{P[B^c]} & x \in B^c \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & x = -3 \\ 0 & \text{else.} \end{cases}$$

$$E[X|B^c] = \sum_{x \in S_{X|B^c}} x \cdot P_{X|B^c}(x) = -3.$$

Ch. 3 Continuous Random Variables

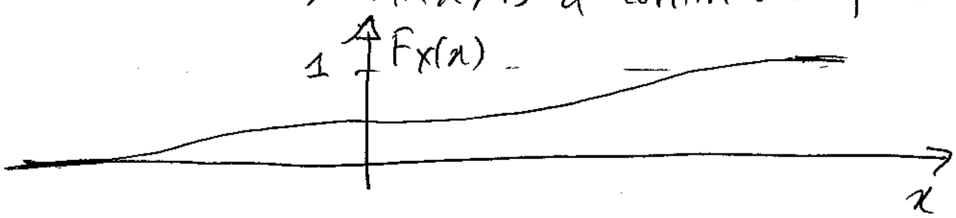
Intuitively, can think of a continuous R.V. X as taking on values in $\mathbb{R}$.

Recall, the CDF is defined as

$$F_X(x_0) \triangleq P[X \le x_0].$$

$\left(F_X(-\infty) = 0, \ F_X(+\infty) = 1, \ F_X(x_1) \le F_X(x_2) \text{ if } x_1 < x_2 \right)$

For a continuous R.V. X, $F_X(x)$ is a continuous function of x.



Properties: $F_X(-\infty) = 0$, $F_X(+\infty) = 1$

$$P[a \le X \le b] = F_X(b) - F_X(a).$$

$F_X(x)$ is a non-decreasing function of x.

Define the probability density function (p.d.f.) as

$$f_X(x) \triangleq \frac{d}{dx} F_X(x).$$

- $f_X(x)$ is a non-negative function (as $F_X(x)$ is a non-decreasing function)
- this means $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du.$

$$P[a < X < b] = \int_a^b f_X(x)\, dx.$$

- $f_X(x)$ is a "probability mass per unit dimension of X"

&

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \qquad \text{(proper density).}$$