Functions of RVs / derived RVs

X : RV, then $Y = g(X)$ is also a RV.

EX: Toss a biased coin with $P[H] = 3/4$. Let X = # heads in one toss.
Then $S_X = \{0, 1\}$, X is Bernoulli with $p = 3/4$.

$$P_X(x) = \begin{cases} 1/4 & \text{for } x = 0 \\ 3/4 & \text{for } x = 1 \\ 0 & \text{else} \end{cases}$$

Let $Y = 100(X+1)^2$: What is S_Y, and what is $P_Y(y)$?

$S_Y = \{100, 400\}$, $\quad P_Y(y) = \begin{cases} 1/4 & y = 100 \\ 3/4 & y = 400 \\ 0 & \text{else.} \end{cases}$

$\cancel{P_X(x) = \{}$

Expected Value of a derived RV

Let $Y = g(X)$, and say we know $P_X(x)$.

Q: What is $E[Y]$? Trick: don't need to calculate $P_Y(y)$!!

$$E[Y] = \sum_{y \in S_Y} y \cdot P_Y(y) = \sum_{y \in S_Y} \boxed{y} \sum_{x : g(x) = y} P_X(x) = \sum_{y \in S_Y} \sum_{x : g(x) = y} \boxed{g(x)} P_X(x)$$

$$E_X[g(X)] = \sum_{x \in S_X} g(x) P_X(x)$$

E.g: Given $Y = g(X) = X(X-1)$. Find $E[Y]$ when X is:

- geometric, binomial, Poisson

Key result: For $0<q<1$

$$T_1 = \sum_{x=0}^{\infty} q^x = \frac{1}{1-q}, \quad \frac{dT_1}{dq} = \sum_{x=1} x q^{x-1} = \frac{1}{(1-q)^2}$$

$$\frac{d^2T_1}{dq^2} = \sum_{x=2}^{\infty} x(x-1) q^{x-2} = \frac{2}{(1-q)^3}$$

$E[X(X-1)]$ wanted. For X geometric, letting $q=1-p$ $(p=P[X=1])$

$$E[Y] = E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\, q^{x-1} p = p \cdot q \sum_{x=0}^{\infty} x(x-1) q^{x-2}$$

$$= p \cdot q \cdot \frac{2}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}.$$

At home, check that X binomial $\Rightarrow E[Y] = n(n-1)p^2$

X Poisson $\Rightarrow E[Y] = \alpha^2$.

Notice: $E[Y] = E[X(X-1)] = E[X^2 - X] = E[X^2] - E[X]$.

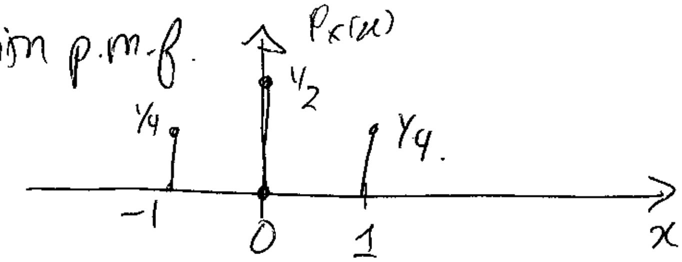
Moments of a RV:

$E[X] \stackrel{DEF}{=} \sum_{x \in S_X} x\, P_X(x) \iff$ "mean", "expected value" "average"

$E[X^2] = \sum_{x \in S_X} x^2 P_X(x) \iff$ "second moment" of R.V. X.

$E[X^m] = \sum_{x \in S_X} x^m P_X(x) \iff$ "m-th moment" of a R.V. X.

E.g: X is a RV with p.m.f.

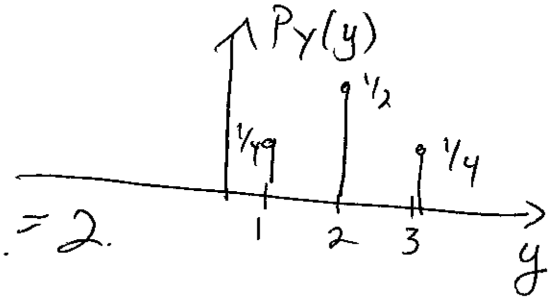


Find the 1st + 2nd moment of X.

$$E[X] = \sum_{x \in S_X} x \cdot P_X(x) = (-1)\tfrac{1}{4} + 0(\tfrac{1}{2}) + 1(\tfrac{1}{4}) = 0$$

$$E[X^2] = \sum_{x \in S_X} x^2 P_X(x) = (-1)^2 \tfrac{1}{4} + 0^2 \tfrac{1}{2} + 1 \cdot \tfrac{1}{4} = \tfrac{1}{2}.$$

Now let $Y = X + 2$. Then $S_Y = \{1, 2, 3\}$



$$E[Y] = E[X+2] = E[X] + E[2] = 0 + 2 = 2.$$
$$= \sum y P_Y(y)$$

$$E[Y^2] = E[(X+2)^2] = E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4.$$
$$= \tfrac{1}{2} + 4 \cdot 0 + 4 = 9/2.$$
$$= \sum_{x \in S_X} (x+2)^2 P_X(x).$$

Special moments: variance and standard deviation.

Variance of a R.V. $\quad Var[X] \triangleq E[(X - \mu_X)^2] \geq 0$

$$\mu_X \triangleq E[X].$$

$$= E[(X - E[X])^2] \geq 0.$$

$\sqrt{Var[X]} \triangleq \sigma_X$ = standard deviation ≥ 0.

<u>Trick</u>: it may be easier (sometimes) to use:

$$Var[X] = E[(X-\mu_X)^2] = E[X^2 - 2X\mu_X + \mu_X^2]$$

$$= E[X^2] - 2\mu_X E[X] + \mu_X^2.$$

$E[K] = K$, $E[K g(X)] = K E[g(X)]$, $E[g(X) + h(X)] = E[g(X)] + E[h(X)]$

<u>EX</u>: What is the variance of a Bernouilli RV?

X = Bernouilli(p), then variance of X is also a function of p !!

$$P_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1-p & \text{for } x = 0 \\ 0 & \text{else} \end{cases}$$

$E[X] = p$. $\quad var[X] = \sum_{x \in \{0,1\}} (x - \mu_X)^2 P_X(x)$

$$= (0-\mu_X)^2(1-p) + (1-\mu_X)^2 p$$

$$= (0-p)^2(1-p) + (1-p)^2 \cdot p$$

$$= p^2 - p^3 + p^2 - 2p^2 + p^3 = p - p^2 = p(1-p)$$