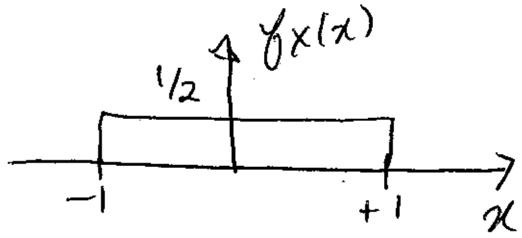
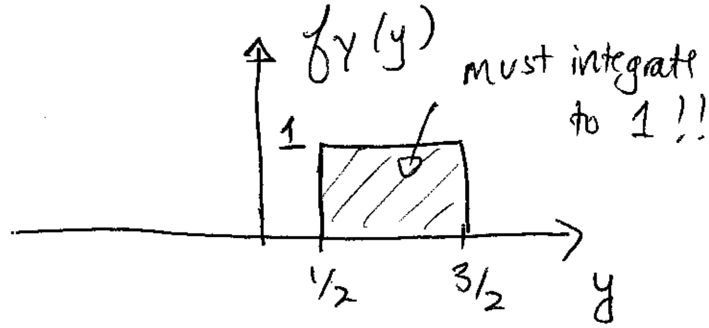
EX: Let X = uniform (continuous) over $[-1, 1]$.

- What is $f_X(x)$?



- $\sigma_X^2 = \frac{(\text{span})^2}{12} = \frac{2^2}{12} = \frac{1}{3}$, std $= \sigma_X = \sqrt{\frac{1}{3}}$, $\mu_X = 0$

- Let $Y = 1 - \frac{X}{2}$.
- What is $f_Y(y)$? $S_Y = [1/2, 3/2]$.



$X = 2(1 - Y)$

$Y = 1 - \frac{X}{2}$

$$\left[\begin{array}{l} P[Y \le y] = P[1 - \frac{X}{2} \le y] \\ = P[X \ge 2(1-y)] = 1 - P[X < 2(1-y)]. \\ F_Y(y) = \cancel{F_X(2(1-y))} = 1 - F_X(2(1-y)) \\ f_Y(y) = \frac{d}{dy} F_Y(y) = -2\,\cancel{f_X(2(1-y))} = 2 f_X(2(1-y)). \end{array}\right]$$

- $\sigma_Y^2 = \frac{(\text{span})^2}{12} = \frac{1^2}{12}$, $\sigma_Y = \sqrt{\frac{1}{12}}$, $\mu_Y = 1$

- What is the correlation coefficient $\rho_{XY} = \frac{\text{cov}(X,Y)}{\text{std}(X)\,\text{std}(Y)} = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$

Need $\text{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$.

$$= E[XY] - \mu_X \mu_Y$$

$$= E[X(1-\tfrac{X}{2})] - 0 \cdot 1 = E[X - \tfrac{X^2}{2}] = E[X] - E[\tfrac{X^2}{2}]$$

$$= 0 - \tfrac{1}{2}E[X^2] = -\tfrac{1}{2}[\sigma_X^2 + \mu_X^2] = -\tfrac{1}{2} \cdot \tfrac{1}{3} = -\tfrac{1}{6}$$

$$\rho_{XY} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}X\,\mathrm{var}Y}} = \frac{-\frac{1}{6}}{\sqrt{\frac{1}{3}\cdot\frac{1}{12}}} = \frac{-\frac{1}{6}}{\frac{1}{6}} = -1$$

$$Y = 1 - \frac{X}{2} \implies \rho_{XY} = -1$$

Theorem: when $Y = aX + b$.

$$\rho_{XY} = \begin{cases} -1 & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ +1 & \text{if } a > 0 \end{cases}$$

Conditioning by an event:

Consider discrete RVs X and Y and an event B with $P[B] > 0$.
Then, the "conditional joint PMF"

"given that B happened"

$$P_{X,Y|B}(x,y) = P[X=x,\ Y=y \mid B]$$

$$\triangleq \begin{cases} \dfrac{P_{X,Y}(x,y)}{P[B]} & \text{if } (x,y) \in B \\ 0 & \text{else} \end{cases}$$

For continuous RVs, we can analagously ~~defined~~ define the conditional joint PDF as

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]} & (x,y) \in B \\ 0 & \text{else.} \end{cases}$$

Conditional expected values :

Conditional expected value of $W = g(X,Y)$ given B is

Discrete $\quad E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) \underbrace{P_{X,Y|B}}_{\text{use conditional distribution!}}(x,y) = \mu_{W|B}$

Continuous $\quad E[W|B] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y|B}(x,y)\, dx\, dy = \mu_{W|B}.$

Conditional variance of $W = g(X,Y)$ given B

$$\mathrm{Var}[W|B] = E\left[(W - \mu_{W|B})^2 \,|\, B\right]$$
$$= E[W^2|B] - (\mu_{W|B})^2$$

E.x.: RVs X, Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5,\ 0 \le y \le 3 \\ 0 & \text{else.} \end{cases}$$

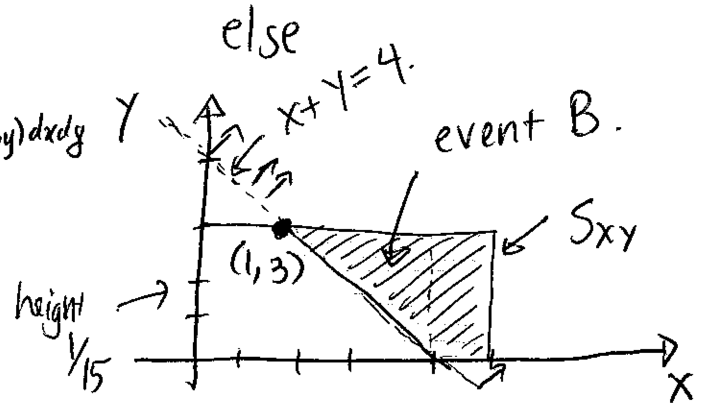
• Find the conditional joint pdf of X and Y given the event $B = \{X+Y \geq 4\}$.

To get $\quad f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]} & (x,y) \in B \\ 0 & \text{else} \end{cases}$

Need $P[B] = P[X+Y \geq 4] = \iint_B f_{XY}(x,y)\,dx\,dy$

$$= \int_{y=0}^{3} \int_{x=4-y}^{5} \frac{1}{15}\, dx\, dy = \frac{1}{2}.$$

check



Then,

$$f_{X,Y|B}(x,y) = \begin{cases} 2/15 & \text{if } 0 \le x \le 5,\ 0 \le y \le 3,\ x+y \ge 4 \\ 0 & \text{else.} \end{cases}$$

EX: Toss a coin and roll a die.

Sample space: $\{h1, h2, h3, h4, h5, h6, t1, t2, t3, t4, t5, t6\}$.

Let X = # heads on cointoss. $S_X = \{0, 1\}$

Y = # dots facing up + # heads on cointoss $\quad S_Y = \{1, 2, 3, 4, 5, 6, 7\}$.