Noisy Communication / Communication over an additive Gaussian noise channel

Transmit samples — Received Signal

$$X \longrightarrow \oplus \longrightarrow Y = X + N$$

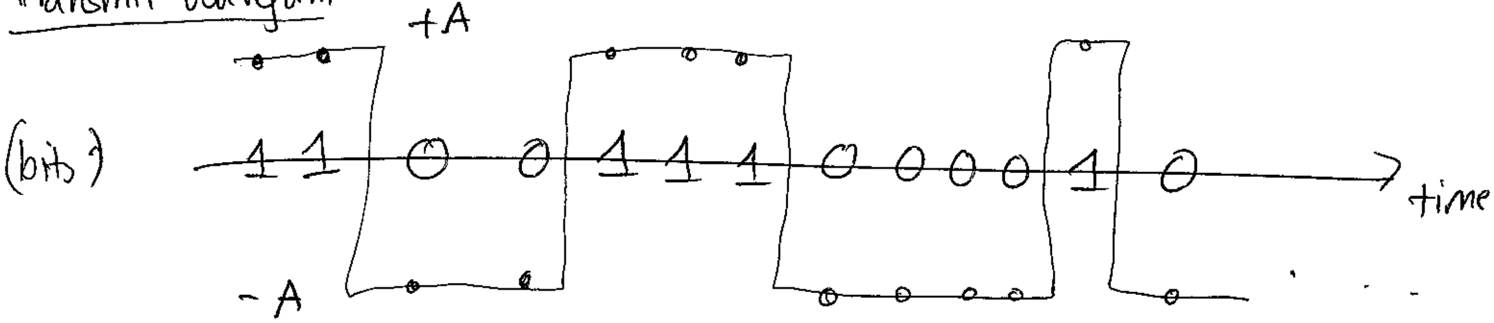
$\uparrow$ Gaussian Noise

$X \in \{-A, A\}$

$P[X=-A] = P[X=A] = 1/2.$

Gaussian Noise modeled as. R.V. $N \sim \mathcal{N}(0, \sigma^2)$

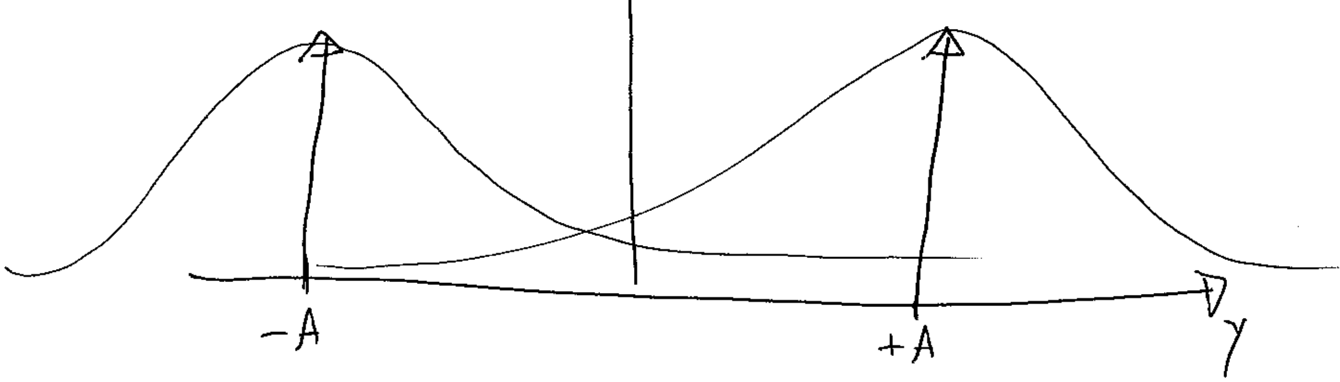
Transmit waveform



Received waveform (after passing through additive Gaussian noise channel):

Received pdf $f_Y(y)$:

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x+A)^2}{2\sigma^2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-A)^2}{2\sigma^2}}$$



$$f_Y(y) = f_{Y|X=-A}(y)\, P[X=-A] + f_{Y|X=+A}(y)\, P[X=+A].$$

Note: $E[X^2] = P[X=-A](-A)^2 + P[X=+A](A)^2$
$= \frac{1}{2}A^2 + \frac{1}{2}A^2 = A^2.$

$$E[N^2] = var(N) + \mu_N^2 = \sigma^2 + 0^2 = \sigma^2$$

Define the Signal-to-Noise Ratio (SNR) of this channel as

$$SNR \triangleq \frac{E[X^2]}{E[N^2]} = \frac{\text{signal power}}{\text{noise power}} \quad \text{or} \quad SNR\,(dB) = 10\log_{10}\left(\frac{E[X^2]}{E[N^2]}\right)\,(dB)$$

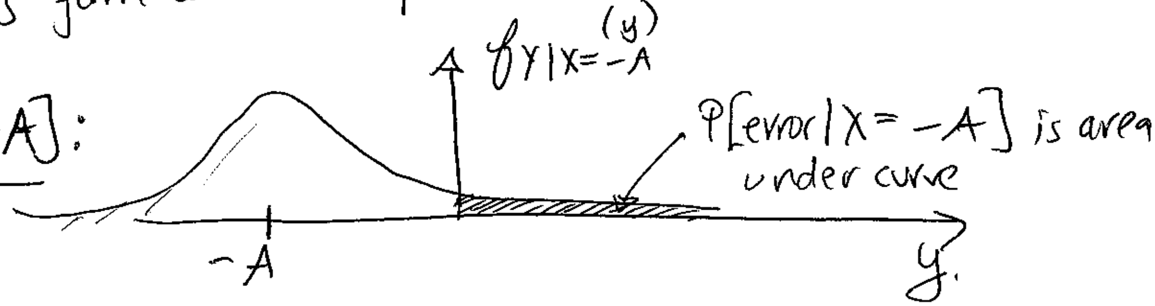
In this case, $SNR(dB) = 10\log_{10}\left(\frac{A^2}{\sigma^2}\right) = 20\log_{10}\left(\frac{A}{\sigma}\right)$

Question: what is the probability of incorrectly estimating the bits if your decision rule: If $Y > 0 \Rightarrow$ decide $+A$ was sent ("1" was sent)
If $Y \leq 0 \Rightarrow$ decide $-A$ was sent ("0" was sent")

$$^*P[error] = P[error \mid X=-A]\,P[X=-A] + P[error \mid X=+A]\,P[X=+A]^*.$$

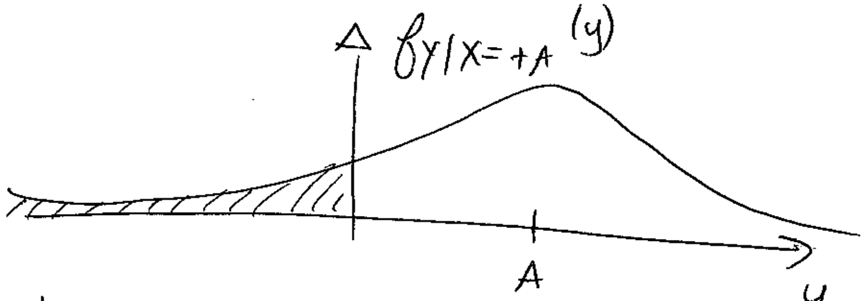
as $\{-A, A\}$ form an event space.

Case $P[error \mid X=-A]$:



$$P[error \mid X=-A] = P[Y>0 \mid X=-A] = P[X+N>0 \mid X=+A] = P[N>+A]$$
$$= P\Big[\underbrace{\frac{N}{\sigma}}_{\sim N(0,1)} > \frac{+A}{\sigma}\Big] = Q\left(\frac{A}{\sigma}\right)$$ just a number! Given A, σ

Case $P[\text{error} \mid X = +A]$:



$$P[\text{error}|X=+A] = P[Y \le 0 \mid X=+A] = P[X+N \le 0 \mid X=+A]$$
$$= P[N \le -A] = P\left[\frac{N}{\sigma} \le \frac{-A}{\sigma}\right] = \Phi\left(\frac{-A}{\sigma}\right) = Q\left(\frac{A}{\sigma}\right)$$

Summary: $X = \{\pm A\}$, $N \sim \mathcal{N}(0, \sigma^2)$ yields $Y = X+N$ will be received incorrectly as

$$P[\text{error}] = P[\text{error}|X=-A]\,P[X=-A] + P[\text{error}|X=+A]\,P[X=+A]$$
$$= \frac{1}{2}Q\left(\frac{A}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A}{\sigma}\right) = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{SNR}\right)$$

Since $SNR = \dfrac{E[X^2]}{E[N^2]} = \dfrac{A^2}{\sigma^2} \Rightarrow$

Ch.4: Two Random Variables

We now examine experiment whose outcomes are associated with two RVs, X and Y, which may or may not be related.

EX: Pick a student at random from those registered in the summer and observe:

- Find $X \in \{1,2,3,4\}$ for the students standing (1="freshman", 2="sophmore", 3="junior", 4="senior")
- Find $Y \in \{1,2\}$ for the # courses the student is registered for.

Consider the following individual PDFs for X and Y:

$$P_X(x) = \begin{cases} 0.15 & x=1 \\ 0.25 & x=2 \\ 0.3 & x=3 \\ 0.3 & x=4 \\ 0 & \text{else} \end{cases} \qquad P_Y(y) = \begin{cases} 0.75 & y=1 \\ 0.25 & y=2 \\ 0 & \text{else.} \end{cases}$$

Question: What is the joint description of X and Y?

"Joint pdf" $P_{XY}(x,y) = P[X=x, \; Y=y]$.

"AND"

Important: Given individual pdfs $P_X(x)$ and $P_Y(y)$, multiple joint distributions $P_{XY}(x,y)$ with those marginals exist!!

One possible joint distribution: all sophmores take exactly 2 course, all others take 1 course.

Y \ X	1	2	3	4
2	0	0.25	0	0
1	0.15	0	0.3	0.3

$= P_{XY}(x,y)$ The joint PDF which must satisfy

$$\sum_{x \in S_X} \sum_{y \in S_Y} P_{XY}(x,y) = 1.$$

Anothe possible joint pdf:

Y \ X	1	2	3	4
2	0	0.05	0.1	0.1
1	0.15	0.2	0.2	0.2

$\equiv P_{XY}(x,y)$ the joint pdf.