EX: R.V. has p.d.f. $f_X(x)$ given by:

$$f_X(x) = \begin{cases} Kx^3 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases} \qquad (K \text{ constant})$$

- Find K.: To ensure a proper p.d.f. $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$.

$$\int_0^1 Kx^3\,dx = \frac{Kx^4}{4}\Big|_0^1 = \frac{K}{4} \overset{\text{WANT}}{=} 1 \implies \boxed{K = 4.}$$

- Find the C.D.F., $F_X(x)$:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \begin{cases} 0 & x < 0 \\ \int_0^x 4u^3\,du & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} = \begin{cases} 0 & x < 0 \\ x^4 & 0 \le x \le 1 \\ 1 & x > 1. \end{cases}$$

- Find $P[-0.5 \le X \le 0.5]$.

$$P[-0.5 \le X \le 0.5] = \int_{-0.5}^{0.5} f_X(u)\,du. = \int_0^{0.5} 4u^3\,du = u^4\Big|_0^{0.5} = (0.5)^4$$

$$= F_X(0.5) - F_X(-0.5) = (0.5)^4 - 0.$$

Statistics of a continuous RV

X is RV. with p.d.f. $f_X(x)$.

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx \qquad \text{"mean"}$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx. \qquad \text{"2nd moment"}$$

$$\mathrm{Var}[X] = E[(X-\mu_X)^2] = \int_{-\infty}^{+\infty} (x-\mu_X)^2 f_X(x)\, dx. \qquad \text{"variance of } X\text{"}$$

$$= E[X^2] - (E[X])^2 = E[X^2] - \mu_X^2.$$

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\, dx \qquad \text{"n-th moment"}$$

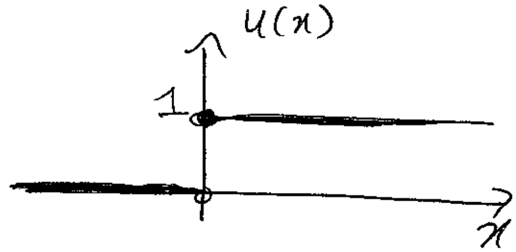
$$E[(X-\mu_X)^n] = \int_{-\infty}^{+\infty} (x-\mu_X)^n f_X(x)\, dx \qquad \text{"n-th central moment"}$$

Recall: standard deviation $\sigma_X \triangleq \sqrt{\mathrm{Var}[X]}$

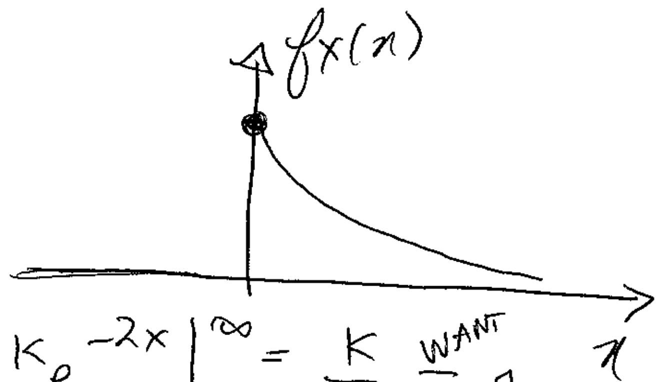
EX: Continuous RV X with $f_X(x) = Ke^{-2x}u(x)$.

- K constant, $u(x) =$ "unit step function"

$$= \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



- $S_X = \{x \mid x \geq 0\}$.



- What is K? Need $\int_{-\infty}^{+\infty} f_X(x) = 1$.

$$\int_{-\infty}^{+\infty} Ke^{-2x}u(x)\,dx = \int_0^{\infty} Ke^{-2x}\,dx = -\frac{K}{2}e^{-2x}\Big|_0^{\infty} = \frac{K}{2} \overset{\text{WANT}}{=} 1$$

$$\Rightarrow \boxed{K=2}$$

- What is $P[X \leq 0.5]$ $= \int_{-\infty}^{0.5} f_X(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^{0.5} 2e^{-2x}\,dx$

$$= e^{-2x}\Big|_0^{0.5} = 1 - e^{-1}$$

- What is $F_X(x)$? $= \int_{-\infty}^{x} f(u)\,du = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x} & x \geq 0 \end{cases}$

- $E[X]$? $= \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^{\infty} x \cdot 2e^{-2x}\,dx \overset{(2x=y)}{=} \frac{1}{2}\int_0^{\infty} ye^{-y}\,dy = \frac{1}{2} \cdot 1!$

$$= \frac{1}{2}.$$

$$\boxed{\text{Useful integral: } \int_0^{\infty} y^n e^{-y}\,dy = n!}$$

Repeated integration by parts to prove

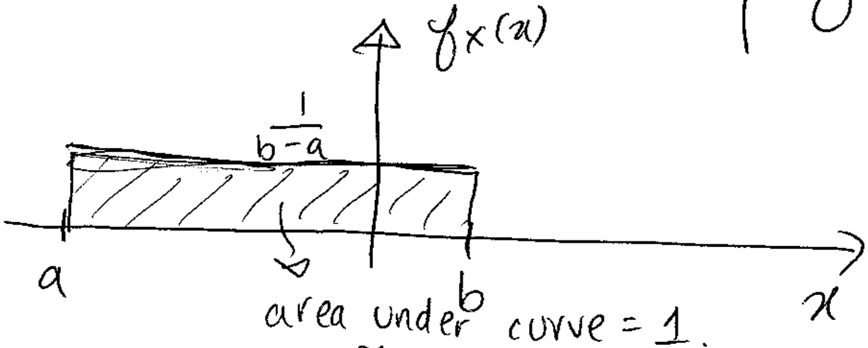
• $E[X^2] := \int_0^\infty x^2 2e^{-2x}\,dx \underset{(2x=y)}{=} \frac{1}{4}\int_0^\infty y^2 e^{-y}\,dy = \frac{1}{4}\cdot 2! = \frac{1}{2}.$

• Standard deviation: $\sigma_X = \sqrt{Var(X)} = \sqrt{E[X^2] - (E[X])^2}$

$$= \sqrt{\frac{1}{2} - \frac{1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Commonly used continuous random variables

Uniform R.V $U(a,b)$: $f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}$

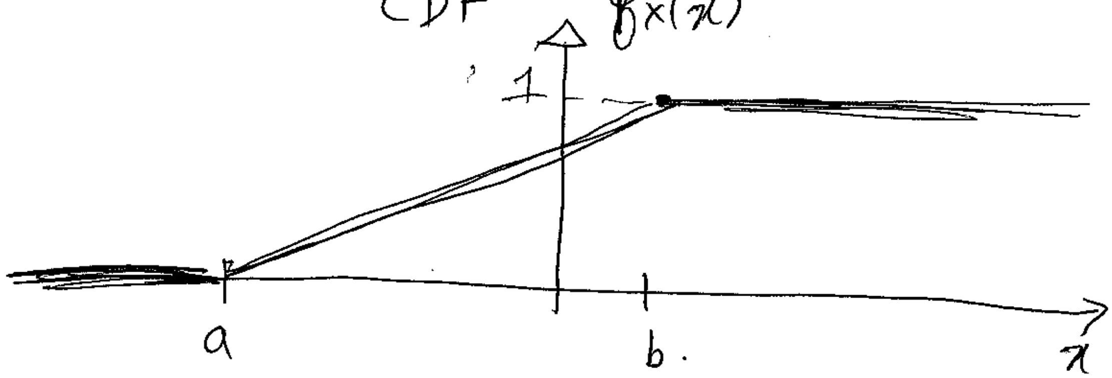


$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x \cdot \frac{1}{b-a}\,dx = \frac{1}{(b-a)}\frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{a^2+ab+b^2}{3}$$

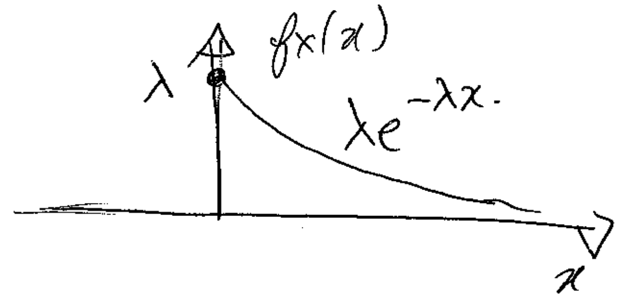
$$Var[X] = E[X^2] - \mu_X^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

CDF



$$F_X(x) = \int_a^x \frac{1}{b-a}\,du = \frac{u}{b-a}\Big|_a^x = \frac{x}{b-a} - \frac{a}{b-a} = \frac{x-a}{b-a}$$

Exponential RV :



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$= \lambda e^{-\lambda x} u(x).$$

$(y = \lambda x.)$

$$\mu_X = E[X] = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \int_0^\infty y e^{-y} dy = \frac{1}{\lambda} \cdot 1! = \frac{1}{\lambda}$$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y} dy = \frac{2}{\lambda^2}.$$

$$\mathrm{Var}[X] = \sigma_X^2 = E[X^2] - \mu_X^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$