Moment generating functions

For a RV X, the moment generating function (MGF) of X is

$$\phi_X(s) = E_X[e^{sX}]$$

$$= \int_{-\infty}^{+\infty} e^{sx} f_X(x)\, dx. \quad \text{for X continous}$$

$$= \sum_{x_i \in S_X} e^{sx_i} P_X(x_i) \quad \text{for X discrete.}$$

Why is called moment generating function?

If we know $\phi_X(s)$, will be easy to calculate the moments of X: $E[X]$, $E[X^2]$, $E[X^3]$, . . . , $E[X^n]$.

Theorem: A RV X with MGF $\phi_X(s)$ has n-th moment

$$E[X^n] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0}$$

This means . . . .

$$E[X] = \left.\frac{d}{ds}\phi_X(s)\right|_{s=0}, \quad E[X^2] = \left.\frac{d^2}{ds^2}\phi_X(s)\right|_{s=0} \cdots$$

Usually, easier to find moments of X by differentiating the MGF than integrating $\int_{-\infty}^{+\infty} x^n f_X(x)\, dx = E[X^n]$

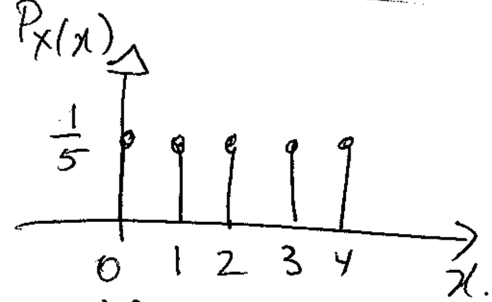
Theorem: What is MGF of $Y = aX + b$? (in terms of a, b, MGF of X)

$$\phi_Y(s) = E_Y\left[e^{sY}\right] = E_X\left[e^{s(aX+b)}\right] = E_X\left[e^{sb}e^{saX}\right]$$

$$= e^{sb}\, E_X\left[e^{saX}\right] = e^{sb}\,\phi_X(as)$$

Quiz 6.3: RV X has pmf

$$P_X(x) = \begin{cases} 0.2 & k = 0,1,2,3,4 \\ 0 & \text{else} \end{cases}$$



Use the MGF to find 1st, 2nd, 3rd, 4th moments of X.

1) Find the MGF of X: $\phi_X(s) = \sum_{x=0}^{4} e^{sx} P_X(x)$

$$= \frac{1}{5}\left[\overset{1}{e^{s\cdot 0}} + e^{s\cdot 1} + e^{s\cdot 2} + e^{s\cdot 3} + e^{s\cdot 4}\right]$$

$$= \frac{1}{5}\left[\frac{1-e^{5s}}{1-e^{s}}\right].$$

2) We know $E[X] = 2$. Let's see if we get this using MGF.

$$E[X] = \frac{d}{ds}\phi_X(s)\Big|_{s=0} = \cancel{\frac{d}{ds}}\left[\frac{1}{5}\left(e^{s} + 2e^{2s} + 3e^{3s} + 4e^{4s}\right)\right]_{s=0}$$

$$= \frac{1}{5}\left[1 + 2 + 3 + 4\right] = 2. \quad \text{as expected!}$$

$$E[X^2] = \frac{d^2}{ds^2}\phi_X(s)\Big|_{s=0} = \frac{d}{ds}\left[\frac{1}{5}\left(e^s + 2e^{2s} + 3e^{3s} + 4e^{4s}\right)\right]\Big|_{s=0}$$ 3

$$= \frac{1}{5}\left[e^s + 2^2 e^{2s} + 3^2 e^{3s} + 4^2 e^{4s}\right]\Big|_{s=0}$$

$$= \frac{1}{5}[1 + 4 + 9 + 16] = 6$$

$$= \sum_{x=0}^{4} x^2 P_X(x) = \frac{1}{5}\left[0^2 + 1^2 + 2^2 + 3^2 + 4^2\right]$$ same !!

... etc. for $E[X^3]$, $E[X^4]$ ....

MGF is good for 2 things:
1) Calculating moments via derivatives
2) Calculating pdf of sums of RVs.

<u>MGF of Sum of Independent RVs</u>

If X, Y are <u>independent</u> RVs with MGFs $\Phi_X(s), \Phi_Y(s)$, what is the MGF of $W = X + Y$?

$$\Phi_W(s) \overset{\text{DEF.}}{=} E_W[e^{sW}] = E[e^{s(X+Y)}] = E[e^{sX}e^{sY}]$$

INDEPENDENT!

$$= E[e^{sX}]E[e^{sY}] = \Phi_X(s)\cdot\Phi_Y(s).$$

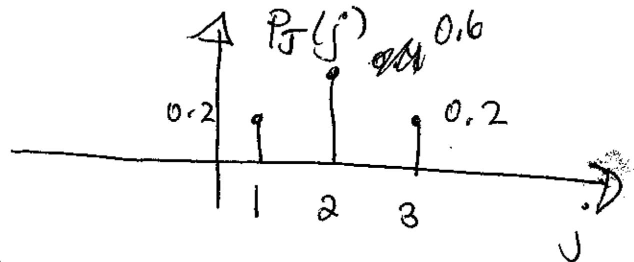
Theorem: $X_1, X_2, \ldots, X_n$ independent random variables, Then the MGF of $W = X_1 + X_2 + \cdots + X_n$ is

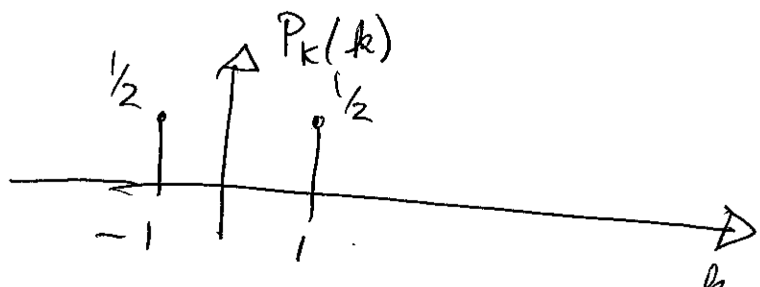
$$\Phi_W(s) = \Phi_{X_1}(s)\,\Phi_{X_2}(s) \cdots \Phi_{X_n}(s).$$

When X_i are i.i.d., Then $X_i \sim X$ with MGF $\Phi_X(s)$

$$\Phi_W(s) = \left[\Phi_X(s)\right]^n.$$

EX: Let J, K be independent RVs with PMFs:





1) Find the MGF of $M = J + K$.

The MGF of $M = J+K$ is

$$\Phi_M(s) = \underbrace{\Phi_J(s)} \cdot \underbrace{\Phi_K(s)} \qquad \text{(as } K, J \text{ independent)}.$$

$$= \left(E_J[e^{sJ}]\right)\left(E_K[e^{sK}]\right)$$

$$= \left(0.2e^{s} + 0.6e^{2s} + 0.2e^{3s}\right)\left(0.5e^{-s} + 0.5e^{s}\right)$$

$$= 0.1 + 0.3e^{s} + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$$

2) What is $P_M(m)$ The pmf of M?

$$= E[e^{sM}].$$

$P(M=0) = 0.1$, $P(M=1) = 0.3$, $P(M=2) = 0.2$, $P(M=3) = 0.3$, $P(M=4) = 0.1$

3) What is $E[m^3]$?

$$E[m^3] = \frac{d^3}{ds^3}\left[\Phi_m(s)\right]\Big|_{s=0}$$

$$= 0.3e^{s} + 0.2(2)^3 e^{2s} + 0.3(3)^3 e^{3s} + (0.1)(4)^3 e^{4s}\Big|_{s=0}$$

$$= 0.3 + 0.2(8) + 0.3(27) + 0.1(64)$$

$$= 16.4$$

Theorem: (can prove these using MGF)

1) $K_1, K_2, \ldots, K_n$ are independent Poisson RVs with ~~means~~ parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$

$\Rightarrow W = K_1 + K_2 + \cdots + K_n$ is also Poisson with parameter $\alpha_W = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

2) $X_1, X_2, \ldots, X_n$ are independent Gaussian with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$

$\Rightarrow W = X_1 + X_2 + \cdots + X_n$ is also Gaussian

with $\mu_W = \mu_1 + \mu_2 + \cdots + \mu_n$ and

$\sigma_W^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$.

Prof ~~Prof~~ part 2:

TABLE

$$X_i \sim N(\mu_i, \sigma_i^2) \Rightarrow \Phi_{X_i}(s) = e^{\left(s\mu_i + \sigma_i^2 \frac{s^2}{2}\right)}$$

$$\Rightarrow \quad \Phi_W(s) = \Phi_{X_1}(s)\,\Phi_{X_2}(s) \cdots \Phi_{X_n}(s)$$ as $X_1, X_2, \ldots, X_n$ independent.

$$= e^{s\mu_1 + \sigma_1^2 \frac{s^2}{2}}\, e^{s\mu_2 + \sigma_2^2 \frac{s^2}{2}} \cdots e^{s\mu_n + \sigma_n^2 \frac{s^2}{2}}$$

$$= e^{\left[s\mu_1 + \sigma_1^2 \frac{s^2}{2} + s\mu_2 + \sigma_2^2 \frac{s^2}{2} + \cdots + s\mu_n + \sigma_n^2 \frac{s^2}{2}\right]}$$

$$= e^{s[\mu_1 + \cdots + \mu_n] + \frac{s^2}{2}[\sigma_1^2 + \cdots + \sigma_n^2]}$$

$$\downarrow$$

This is MGF of $N(\mu_1 + \mu_2 + \cdots + \mu_n,\ \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)$.