

ECE 341: Probability and Random Processes for Engineers, Spring 2012

Homework 10

Name:

Assigned: 03.28.2012

Due: 04.04.2012

Problem 1. Textbook problem 5.4.5. Do it on your own rather than looking at the solution.

Solution 1: Evaluate the joint density function, it is not equal to the product of marginal pdfs then they are not independent

$$f_{X_1, X_2, X_3}(10, 9, 8) = 0 \neq f_{X_1}(10)f_{X_2}(9)f_{X_3}(8) \quad (1)$$

Problem 2. Textbook problem 5.6.1. Do it on your own rather than looking at the solution.

Solution 2:

$$C_X = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \quad (2)$$

b- - Let $Y = [X_1 \ X_2]'$ since Y is linear combination of gaussian random variables then it is also a gaussian vector.

$$C_Y = AC_X A^T = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix} \quad (3)$$

Problem 3. Textbook problem 5.7.1. Do it on your own rather than looking at the solution.

Solution 3: The correlation matrix is

$$R_x = C_x + \mu_x \mu_x' = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix} \quad (4)$$

b- Let $Y = [X_1 \ X_2]'$ since Y is subset of gaussian then it is also a gaussian vector

$$E(Y) = [E(X_1) \ E(X_2)]' = [4 \ 8]' \quad (5)$$

$$C_Y = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \quad (6)$$

(7)

$$C_Y^{-1} = \frac{1}{20} \begin{bmatrix} 4 & +2 \\ +2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad (8)$$

(9)

Now solving for the exponent $(Y - \mu_Y)K_Y^{-1}(Y - \mu_Y)$

$$\begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix} = \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16y_1}{3} - \frac{20y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3} \quad (10)$$

The PDF of Y is

$$f_Y(y) = \frac{1}{\sqrt{48\pi^2}} e^{\left(-\frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16y_1}{3} - \frac{20y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3}\right)} \quad (11)$$

c- With the use of Q- function

$$P(X_1 > 8) = Q(2) = 0.0288 \quad (12)$$

Problem 4. Suppose X and Y are jointly Gaussian distributed with parameters $E[X] = \mu_x$, $E[Y] = \mu_y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$ and $E[XY] = \rho_{XY}$. Are the following statements true or false?

1. X and Y are then also Gaussian with $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.
2. If $\rho_{XY} = 0$ then X and Y are independent.
3. $Z = aX + bY$ for known constants a, b is Gaussian with mean $a\mu_X + b\mu_Y$.
4. $Z = aX + bY$ for known constants a, b is Gaussian with variance $a^2\sigma_X^2 + b^2\sigma_Y^2$.
5. The linear minimum mean squared estimator of X in terms of Y is $\hat{X} = \mu_X + \rho_{XY}(Y - \mu_Y)\frac{\sigma_X}{\sigma_Y}$.
6. The conditional distribution of $Y|X$ is Gaussian.
7. $E[X|Y] = \mu_X + \rho_{XY}(Y - \mu_Y)\frac{\sigma_X}{\sigma_Y}$.

Solution 4:

1. X and Y are then also Gaussian with $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.
Yes, this is true. A joint Gaussian distribution implies marginally distributed gaussian.
2. If $\rho_{XY} = 0$ then X and Y are independent.
 $\rho_{XY} = 0$ then $Cov(X, Y) = 0$ and $E(XY) = E(X)E(Y)$ then X and Y are uncorrelated and in the case of gaussian only uncorrelated random variables implies independent.
3. $Z = aX + bY$ for known constants a, b is Gaussian with mean $a\mu_X + b\mu_Y$.
This is true, $E(Z) = aE(X) + bE(Y) = a\mu_X + b\mu_Y$
4. $Z = aX + bY$ for known constants a, b is Gaussian with variance $a^2\sigma_X^2 + b^2\sigma_Y^2$.
This is not true, since we did not account for the covariance between X and Y .
 $Var(Z) = 2ab Cov(X, Y) + Var(X) + Var(Y)$

5. The linear minimum mean squared estimator of X in terms of Y is $\hat{X} = \mu_X + \rho_{XY}(Y - \mu_Y)\frac{\sigma_X}{\sigma_Y}$. True. This is the optimal estimator.
6. The conditional distribution of $Y|X$ is Gaussian. True, Conditioning preserves Gaussianity.
7. $E[X|Y] = \mu_X + \rho_{XY}(Y - \mu_Y)\frac{\sigma_X}{\sigma_Y}$. True in fact this is the Linear minimum mean square estimate of X .

Problem 5. Suppose X and Y are zero-mean unit-variance jointly Gaussian random variables with correlation coefficient $\rho = 0.5$.

1. Find $Var(3X - 2Y)$.
2. Find the numerical value of $P[(3X - 2Y)^2 \leq 28]$.
3. Find the numerical value of $E[Y|X = 3]$.

Solution 5:

$$Var(3X - 2Y) = Cov(3X - 2Y, 3X - 2Y) = 9Var(X) + 4Var(Y) - 12Cov(XY) = 9 - 6 + 4 = 7 \quad (13)$$

Find the numerical value of $P[(3X - 2Y)^2 \leq 28]$.

Define a Random variable $Z = 3X - 2Y$. Keeping in mind that any linear combination of Gaussian is Gaussian then Z is a Gaussian with $E(Z) = 0$ and $Var(Z) = 7$

$$P(Z^2 \leq 28) = P(-\sqrt{28} \leq Z \leq \sqrt{28}) = \Phi\left(\frac{\sqrt{28}}{7}\right) - \Phi\left(\frac{-\sqrt{28}}{7}\right) = 2(\Phi(2) - 0.5) = 0.9545. \quad (14)$$

$$E[Y|X = 3] = \mu_Y + (x - \mu_x)\frac{Cov(XY)}{Var(X)} \quad (15)$$

$$= 0 + (3)\rho_{X,Y} = 1.5 \quad (16)$$