ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 9
Name:
Assigned: 03.07.2012
Due: 03.14.2012

Problem 1. Textbook problem 4.10.11. Do it on your own rather than looking at the solution.

Solution 1:
a- In order to find the Joint PDF $P_{B,M}(b,m)$. The key is to understand that Factory Q and Factory R are associated with $M = 60$ and $R = 180$. And similarly the orders: small, medium and large are associated with $B = 1, B = 2, B = 3$. Thus the table given in the exercise represents the Joint PDF .

$P_{B,M}(b,m)$	$m = 60$	$m = 180$
$b = 1$	0.3	0.2
$b = 2$	0.1	0.2
$b = 3$	0.1	0.1

b- In order to calculate the expected value $E[B]$, we need to calculate the marginal PDF i.e $P_B(b)$. The table below shows the marginal PDFs of both $P_B(b)$ and $P_M(m)$.

$$E(B) = \sum_b (b) P_B(b) = 1(0.5) + 2(0.3) + 3(0.2) = 1.7 \tag{1}$$

c- From Part b we saw that the marginal PDF $P_B(b = 2) = 0.3$. The conditional PMF of M given B=2 is

$$P_{M|B=2}(m|b = 2) = \frac{P_{M,B=2}(m, 2)}{P_B(b = 2)} \tag{2}$$

$$= \begin{pmatrix} \frac{1}{3} & m = 60 \\ \frac{2}{3} & m = 180 \\ 0 & \text{otherwise} \end{pmatrix}$$

$P_{B,M}(b,m)$	$m=60$	$m=180$	$P_B(b)$
$b=1$	0.3	0.2	0.5
$b=2$	0.1	0.2	0.3
$b=3$	0.1	0.1	0.2
$P_M(m)$	0.5	0.5	

d- The conditional expectation of M given B =2 is

$$E(M|B=2) = \sum_m m P_{M|B}(m|B=2) = 60(\frac{1}{3}) + \frac{2}{3}120 = 140 \tag{3}$$

e- From the marginal PDFS we calculated it is clear that the two RV are not independent since

$$P_{BM}(b=1, m=60) \neq P_B(b=1)P_M(m=60) \tag{4}$$

Problem 2. Show the following identities, for X, Y, U, V random variables which are not necessarily independent, and a, b, c, d are known constants.

- $\text{Cov}(X+Y, U+V) = \text{Cov}(X,U) + \text{Cov}(X,V) + \text{Cov}(Y,U) + \text{Cov}(Y,V)$.
- $\text{Cov}(aX+b, cY+d) = ac\text{Cov}(X,Y)$.

Solution 2:

$$\begin{aligned} a - \text{Cov (X+Y, U +V)} &= E[(X+Y)(U+V)] - E(X+Y)E(U+V) \qquad (5) \\ &= E[XU + XV + YU + YV] - E(X+Y)E(U+V) \\ &= E(XU) + E(XV) + E(YU) + E(YV) - [E(X)+E(Y)][E(U)+E(V)] \\ &= E(XU) + E(XV) + E(YU) + E(YV) - E(X)E(U) - E(X)E(V) \\ &- E(Y)E(U) - E(Y)E(V) \\ &= E(XU) - E(X)E(U) + E(XV) - E(X)E(V) + E(YV) - E(Y)E(V) \\ &+ E(YU) - E(Y)E(U) \\ &= \text{Cov}(X,U) + \text{Cov}(X,V) + \text{Cov}(Y,U) + \text{Cov}(Y,V) \end{aligned}$$

b-

$$\begin{aligned}
\text{Cov}(aX+b, cY+d) &= E[(aX+b)(cY+d)] - E(aX+b)E(cY+d) \qquad (6)\\
&= E(acXY + aXd + bcY + bd) - [aE(X)+b][cE(Y)+d]\\
&= acE(XY) + adE(X) + bcE(Y) + bd - acE(X)E(Y)\\
&- adE(X) - bcE(Y) - bd\\
&= acE(XY) - acE(X)E(Y)\\
&= acCov(X,Y)
\end{aligned}$$

Problem 3. Let $X = Y + N$, where Y has the exponential distribution with parameter λ and N is Gaussian with mean 0 and variance σ^2. Suppose the variables Y and N are independent and the parameters $\lambda > 0$ and $\sigma^2 > 0$ are known. Find the mean-squared error linear estimator of Y given X.

Solution 3: The Linear Mean Square estimate of Y given X is $\hat{Y} = aX + b$

$$E(Y|X=x) = \mu_Y + \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x - \mu_X) \qquad (7)$$

What we need to do next is calculate the values of μ_X, μ_Y, ρ_{XY}

$$\mu_X = \frac{1}{\lambda} \text{ Property of exponential distribution} \qquad (8)$$

$$\mu_Y = \mu_X + \mu_N = \frac{1}{\lambda} \qquad (9)$$

$$Cov(Y,X) = Cov(Y, Y+N) = Cov(Y,Y) + Cov(Y,N) = \sigma_Y^2 + (\frac{1}{\lambda})^2 \qquad (10)$$

$$\rho_{XY} = \frac{Cov(Y,X)}{\sigma_X\sigma_Y} \qquad (11)$$

$$\sigma_X^2 = \sigma_Y^2 + \sigma_N^2 = (\frac{1}{\lambda})^2 + \sigma^2 \qquad (12)$$

$$\qquad (13)$$

Thus

$$E(Y|X=x) = \mu_Y + \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x-\mu_X) \quad (14)$$

$$= \mu_Y + \frac{Cov(X,Y)}{\sigma_X}\sigma_X(x-\mu_X) \quad (15)$$

$$= \frac{1}{\lambda} + \frac{(\frac{1}{\lambda})^2}{(\frac{1}{\lambda})^2+\sigma^2}(X-\frac{1}{\lambda}) \quad (16)$$

Problem 4. Textbook problem 4.11.3. Do it on your own rather than looking at the solution.

Solution 4: From the problem statement we learn that

$$\mu_X = \mu_Y = 0 \quad (17)$$

$$\sigma_X^2 = \sigma_Y^2 = 1 \quad (18)$$

$$(19)$$

The conditional expectation of Y given X is

$$E(Y|X=x) = \mu_Y + \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x-\mu_X) \quad (20)$$

$$= \rho_{XY}x \quad (21)$$

From the problem it is seen that $X = \frac{Y}{2}$, thus $\rho_{XY} = 0.5$. The joint PDF is

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{3\pi^2}}e^{\frac{-2(x^2-xy+y^2)}{3}} \quad (22)$$