ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 8
Name:
Assigned: 02.29.2012
Due: 03.07.2012

Problem 1. Suppose (X, Y) is uniformly distributed (continuous) over the unit circle, i.e. $S_{XY} = \{(x, y) : x^2 + y^2 \leq 1\}$.

1. What is the joint pdf $f_{X,Y}(x, y)$?
2. Find $P[A]$ if A is the event $A = \{(x, y) : u \geq 0, v \geq 0\}$.
3. Find $P[X^2 + Y^2 \leq r^2]$ for $r \geq 0$.
4. Find the marginal pdf of X.
5. Find the conditional pdf of Y given X.

Solution 1: 1. For joint PDF $f_{X,Y}(x, y)$: Keeping in mind that the double integral of the joint PDF should end up equal to one and that the Area of $S = \pi r^2 = \pi$ and that (X, Y) are uniformly distributed. Then the joint PDF is

$$f_{XY}(x) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

2.Find $P[A]$ if A is the event $A = \{(u, v) : u \geq 0, v \geq 0\}$

$P[A]$ is the probability of the event $A = \{(u, v) : u \geq 0, v \geq 0\}$. The region $A \cap S$ specified here is just quarter the Area of the circle. Thus it is $\frac{1}{4}$.

3. Find $P[X^2 + Y^2 \leq r^2]$ for $r \geq 0$.
If $0 \leq r \leq 1$, the region $(x, y) : X^2 + Y^2 \leq r^2$ is a disk of radius r contained in S. The area of this region intersect S. The area of this region πr^2. Dividing this by the area by S yields: $P[X^2 + Y^2 \leq r] = r^2$, for $0 \leq r \leq 1$. If $r > 1$, the region $X^2 + Y^2 \leq r^2$ contains all S so $P[X^2 + Y^2 \leq r] = 1$ for $r > 1$.

4. Find the marginal pdf of X.

$$f_X(x) = \begin{cases} \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dy = \int_{\sqrt{1^2 - x^2}}^{\sqrt{1^2 + x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi} & \text{if} |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

4. Find the conditional pdf of Y given X

$$f_{Y|X}(y|x) = \begin{cases} \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-x^2}}{\pi}} = \frac{1}{2\sqrt{1-x^2}} & \text{if} -\sqrt{1 - x^2} \leq y \leq +\sqrt{1 - x^2} \\ 0 & \text{otherwise} \end{cases}$$

That is if $|x| \leq 1$, then given $X = x$, Y is uniformly distributed over the interval $[-\sqrt{1-x^2}, +\sqrt{1-x^2}]$. This makes sense geometrically a slice through the cylindrically shaped region under the joint pdf is a rectangle.

Problem 2. Suppose one fair die (with six equi-probable sides) is rolled. Let

$$X = \begin{cases} 1 & \text{if "one" shows} \\ 0 & \text{else} \end{cases} \qquad Y = \begin{cases} 1 & \text{if "two" shows} \\ 0 & \text{else} \end{cases}$$

1. Find the marginal pmfs of X and Y.
2. Find $E[X], E[Y], \mathrm{Var}(X), \mathrm{Var}(Y)$.
3. Find the joint pmf of (X, Y), $P_{X,Y}(x, y)$.
4. Find $\mathrm{Cov}(X, Y)$.
5. Find the correlation coefficient between X and Y.

Solution 2:
1. $P_X(x) = \frac{1}{6}$ and $P_Y(y) = \frac{1}{6}$

2. $E(X) = \frac{1}{6}$, $E(Y) = \frac{1}{6}$, $Var(X) = E(X^2) - E(X)^2 = p - p^2 = \frac{5}{36}$,
$Var(Y) = E(Y^2) - E(Y)^2 = p - p^2 = \frac{5}{36}$.

3. Joint pmf of (X, Y), $P_{X,Y}(x, y)$.

$P_{X,Y}(0,0) = \frac{4}{6}, P_{X,Y}(1,0) = \frac{1}{6}, P_{X,Y}(0,1) = \frac{1}{6}$, $P_{X,Y}(1,1) = 0$

4.$\mathrm{Cov}(X, Y)$ $=E(XY) - E(X)E(Y) = 0 - \frac{1}{36} = -\frac{1}{36}$

Note that E(XY) is equal to zero because it is impossible to have an X and Y at same time with the same dice.

5. Find the correlation coefficient between X and Y

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-\frac{1}{36}}{\sqrt{\frac{5}{36} \cdot \frac{5}{36}}} = \frac{-1}{5}. \tag{1}$$

Problem 3. Let $Z = \frac{Y}{X^2}$ where X, Y have the joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{else} \end{cases}.$$

1. Find $P[Z \le 0.5]$.
2. Find $P[Z \le 4]$.

Solution 3: Find $P[Z \le 0.5]$.

$$P[Z \le 0.5] = P[Y \le 0.5X^2] = \int_{x=0}^{1} \int_{y=0}^{0.5x^2} 1 dy\, dx = \int_0^1 0.5x^2 dx = \frac{1}{6} \tag{2}$$

$$P[Z \le 4] = P[Y \le 4X^2] = \int_0^1 4x^2 dx + 0.5 = \frac{2}{3} \tag{3}$$

$$\tag{4}$$

Problem 4. Suppose $W = \max(X, Y)$, where X and Y are independent, continuous-type random variables. Express the pdf of W in terms of the pdfs of X and Y. *(HINT: First find CDF then differentiate)*

Solution 4: Let us start with the CDF $F_W(w)$

$$F_W(w) = P(\max(X,Y) \le w) = P(X \le w)P(Y \le w) = F_X(w)F_Y(w) \tag{5}$$

The Independence of the two Random Variables X and Y and the fact that $\max(X, Y) \le w$ takes place only if $X \le w$ and if $Y \le w$ allows us to write the above.
Now taking the derivative of the above expression with respect to w gives us

$$f_W(w) = f_X(w)F_Y(w) + f_Y(w)F_X(w) \tag{6}$$